PROOF OF THE FINITENESS OF MODULAR COVARIANTS*

BY

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1. Let $f_1, \cdots, f_l$ be any system of forms (homogeneous polynomials) in the arbitrary variables $x_1, \cdots, x_m$ with undetermined integral coefficients taken modulo p, where p is a prime. Let $c_1, c_2, \cdots$ denote the coefficients arranged in any order. Under the transformation

$$T: \qquad x_i \equiv \sum_{j=1}^{m} t_{ij} x_j' \pmod{p} \qquad (i = 1, \cdots, m),$$

with integral coefficients, let f_i become the form f_i', and let $c_1', c_2', \cdots$ denote the coefficients of $f_1', \cdots, f_l'$ corresponding in position to $c_1, c_2, \cdots$, respectively. A rational integral function

$$K(c_1, c_2, \cdots; x_1, \cdots, x_m)$$

with integral coefficients taken modulo p is called a modular covariant of the forms $f_1, \cdots, f_l$ (under the group G of all transformations T) if, for every transformation T,

$$K(c_1', c_2', \cdots; x_1', \cdots, x_m') \equiv |t_{ij}|^{\mu} K(c_1, c_2, \cdots; x_1, \cdots, x_m) \pmod{p}$$

holds identically in $c_1, c_2, \cdots, x_1', \cdots, x_m'$, after $x_1, \cdots, x_m$ are eliminated by means of the congruences T, and $c_1', c_2', \cdots$ are replaced by their expressions in terms of $c_1, c_2, \cdots$. The exponent μ is called the index of K.

As an immediate generalization, we may take the coefficients c_i of the forms $f_1, \cdots, f_l$, the coefficients t_{ij} of the transformations T, and the coefficients of the covariant polynomials $K(c_i; x_j)$ to be Galois imaginaries

$$r_0 + r_1\rho + r_2\rho^2 + \cdots + r_{n-1}\rho^{n-1} \qquad (r_i = 0, 1, \cdots, p-1),$$

in which ρ is a root of a fixed congruence of degree n, irreducible modulo p. The sum, difference, product or quotient (the divisor not being 0) of any two of these p^n Galois imaginaries is one of these same imaginaries, so that they form a field of order p^n. This field is called the Galois field of order p^n and designated $GF[p^n]$. For $n = 1$, the field is that of the integral residues

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modulo p; it is for this case that we gave above the explicit definition of modular covariants.

We do not consider in the present paper covariants of two or more sets of cogredient variables.

Any rational integral algebraic covariant with integral coefficients of a system of algebraic forms f_i becomes a modular covariant of that system of forms when the coefficients of the f_i are interpreted as arbitrary elements of any $GF[p^n]$. We obtain in this way only a relatively small proportion of the modular covariants. This fact is illustrated (§§ 7–13) by a complete set of fundamental covariants of the binary quadratic form in the $GF[3]$. Notwithstanding this greater prolixity of modular covariants, I obtain the

Theorem. *The set of all modular covariants of any system of forms in m variables is finite, in the sense that they are all rational integral functions, with coefficients in the basal field, of a finite number of the covariants of the set.*

For $m = 2$ the proof is very simple (§ 2). For $m \geqq 3$ use is made of a lemma* on the finiteness of any set of monomial functions. The nature of the proof is best seen in the typical case $m = 3$ (§ 3), when the functions and formulæ employed are quite simple and the lemma [the existence of relations (9)] is proved directly (without induction). The essential points in the extension to m variables are given in § 4.

Finiteness of the modular covariants of binary forms.

2. By the theorem on the multiplication of two determinants of the second order, the function

$$\begin{vmatrix} x^{p^{2n}} & y^{p^{2n}} \\ x & y \end{vmatrix}$$

is an invariant of index unity under any linear homogeneous transformation on x, y, with coefficients in the $GF[p^n]$. Hence the group G of all such transformations has the universal rational integral covariants†

$$(1) \qquad L = x^{p^n} y - x y^{p^n}, \qquad Q = \frac{x^{p^{2n}} y - x y^{p^{2n}}}{L} = x^P + \cdots \qquad (P = p^{2n} - p^n),$$

Q being an absolute covariant. We require also the fact that

$$(2) \qquad L = y\Pi (x - ay) \qquad (a \text{ ranging over the } GF[p^n]).$$

* In papers cited in § 4, I have recently employed this lemma to prove that there exist only a finite number of perfect or abundant numbers not multiples of smaller perfect or abundant numbers and having a given number of distinct odd prime factors and a given number of factors 2.

† Dickson, these Transactions, vol. 12 (1911), p. 1. It is there shown, in the spirit of finite projective geometry, that L and Q form a fundamental system of invariants of the group G. While this theorem is not presupposed in the present paper, it affords an explanation of the success attending the use here of L and Q.

A homogeneous covariant K, of order Ω, of a system Σ of binary forms shall be called regular or irregular according as its leader (the coefficients of x^Ω) is not zero or zero. An irregular covariant has the factor y and therefore, by (2), the factor L. The number of leaders S of regular covariants is finite, since S is a polynomial, with coefficients in the $GF[p^n]$, in the coefficients c_i of the forms of the system Σ and each c_i has an exponent less than p^n.

All regular covariants with a given leader S and orders Ω congruent modulo P to a given integer ω shall be said to form the set $[S, \omega]$. Since ω may be restricted to P or fewer values, and the number of leaders S is finite, the number f of sets is finite. Let

$$k = Sx^\omega + \cdots$$

be a covariant of the set $[S, \omega]$ whose order ω is the least of the orders of covariants of this set. Then any new covariant of this set is of the form

$$K = Sx^{\omega + tP} + \cdots \qquad (t \text{ an integer} \geqq 0).$$

Hence $K - kQ^t$ is a covariant with the factor y and hence an irregular covariant. Thus any regular covariant is of the form

$$\phi(k_1, \cdots, k_f, Q) + I, \tag{3}$$

where ϕ is a polynomial with integral coefficients in Q and the covariants $k_1, \cdots, k_f$, one corresponding to each of the f sets $[S, \omega]$, while I is an irregular covariant. Now $I = L^e K'$, where K' is a regular covariant. Expressing K' in the form (3) and treating $I' = L^{e'} K''$ similarly, we ultimately obtain the result that every covariant is a polynomial in $k_1, \cdots, k_f, Q, L$.*

Finiteness of the modular covariants of ternary forms.

3. The determinants, of which only the ith column is written,

$$L_3 = \begin{vmatrix} x_i^{p^{2n}} \\ x_i^{p^n} \\ x_i \end{vmatrix}, \qquad L_3' = \begin{vmatrix} x_i^{p^{2n}} \\ x_i^{p^{2n}} \\ x_i \end{vmatrix}, \qquad L_3'' = \begin{vmatrix} x_i^{p^{2n}} \\ x_i^{p^n} \\ x_i \end{vmatrix} \qquad (i = 1, 2, 3) \tag{4}$$

are invariants of index unity under any linear homogeneous transformation on x_1, x_2, x_3, with coefficients in the $GF[p^n]$. Hence the group G of all such transformations has the universal rational integral† covariants

$$L_3, \qquad Q_{31} = L_3' / L_3, \qquad Q_{32} = L_3'' / L_3, \tag{5}$$

* If the group is a subgroup G' of the total group G, the only modification in the proof is the replacement of L by the product of the non-proportional linear functions into which y is transformed by G'.

† DICKSON, these Transactions, vol. 12 (1911), pp. 76, 77. The three form a fundamental system of invariants of the group G.

Q_{31} and Q_{32} being absolute covariants. By considering the minors of x_3 in the determinants (4), we find at once that, in the field,

$$(6)\quad Q_{32} = x_1^P + \cdots, \quad Q_{31} = L^E + x_3(\,), \quad P = p^{3n} - p^{2n}, \quad E = p^{2n} - p^n,$$

where

$$(7)\qquad L = x_1^{p^n} x_2 - x_1 x_2^{p^n}$$

as in (1).

A homogeneous covariant K of a system Σ of ternary forms f_i is called regular or irregular, according as it is not or is divisible by x_3. Thus an irregular covariant has the factor L_3 (see paper last cited).

Let k be the sum of the terms of K not involving x_3. Under a linear transformation on x_1, x_2 alone, let k become k_1 and f_i become f_i'. By the covariancy of K, k_1 is identical apart from a constant factor with the function k built for the coefficients and variables of the f_i'. Hence if k has the factor x_2 it has the factor L, given by (7). Thus

$$k = L^A (Sx_1^B + S_1 x_1^{B-1} x_2 + \cdots) \qquad (S \neq 0,\ A \geqq 0).$$

We shall call S the leader of the regular covariant K.

Denote by $[S, a, b]$ the set of all the regular covariants

$$(8)\quad K = SL^A x_1^B + \cdots, \qquad A \equiv a \pmod{E}, \qquad B \equiv b \pmod{P},$$

in which S is a given leader $\neq 0$, while a and b are given integers. From this set we may select a finite number of covariants

$$K_i = SL^{A_i} x_1^{B_i} + \cdots \qquad (i = 1, \cdots, t)$$

such that, in any covariant (8) of the set,

$$(9)\qquad A \geqq A_i, \qquad B \geqq B_i \qquad (\text{for some value of } i \leqq t).$$

Indeed, we may take as A_1 the least A, and as B_1 the least B occurring in the $L^{A_1} x_1^B$; as B_2 the least B, and as A_2 the least A occurring in the $L^A x_1^{B_2}$. Hence, for a given covariant (8), relations (9) hold for $i = 1$ or 2, unless both $A < A_2$ and $B < B_1$. But the latter both hold only for a finite number of pairs of integers A, B, each $\geqq 0$, and these may be denoted by A_i, B_i $(i = 3, \cdots, t)$. Hence, by (8), any covariant of the set $[S, a, b]$ has the form

$$K = SL^{A_i + lE} x_1^{B_i + mP} + \cdots,$$

where l, m are integers $\geqq 0$ and i is one of the integers $1, \cdots, t$. Hence, by (6), the terms free of x_3 in the covariant

$$K' = K - K_i Q_{31}^l Q_{32}^m$$

have the factor $L^A x_2$. Thus K' is either irregular or is of the form

$$S' L^{A'} x_1^{B'} + \cdots, \qquad S' \neq 0, \qquad A' > A, \qquad B' < B.$$

Let $K_1, \cdots, K_\tau$ serve for all the sets as did $K_1, \cdots, K_t$ for the particular set $[S, a, b]$. Thus any regular covariant has the form

$$K = K_i Q_{31}^l Q_{32}^m + K' \qquad (i \leqq \tau),$$

where K' is either irregular or else is a regular covariant whose B' is less than the B of K. In the latter case we have

$$K' = K_j Q_{31}^{l'} Q_{32}^{m'} + K'' \qquad (j \leqq \tau),$$

where K'' is either irregular or else is a regular covariant whose B'' is less than B'. Hence, finally,

$$K = f(K_1, \cdots, K_\tau, Q_{31}, Q_{32}) + I, \tag{10}$$

where I is an irregular covariant, and f is a rational integral function with integral coefficients. Now

$$I = L_3^c K_0,$$

where K_0 is a regular covariant, possibly a constant. Expressing K_0 in the form (10), and repeating the process, we ultimately obtain K expressed as a polynomial in $K_1, \cdots, K_\tau, Q_{31}, Q_{32}, L_3$.

Finiteness of the modular covariants of m-ary forms.

4. The determinants of order m

$$L_m = |\, x_i^{p^{(m-1)n}} \cdots x_i^{p^n} x_i \,|, \qquad L_m^{(s)} = |\, x_i^{p^{mn}} \cdots x_i^{p^{(s+1)n}} x_i^{p^{(s-1)n}} \cdots x_i \,| \qquad (i = 1, \cdots, m)$$

are invariants of index unity under the group G of all linear homogeneous transformations on $x_1, \cdots, x_m$ with coefficients in the $GF[p^n]$. The quotients $Q_{ms} = L_m^{(s)} / L_m$ are integral functions in the field and are absolute universal covariants. By considering the minors of x_m, we find at once that

$$L_m^{(1)} / L_m = L_{m-1}^{p^n - p^n} + x_m(), \qquad L_m^{(s)} / L_m = (L_{m-1}^{(s-1)} / L_{m-1})^{p^n} + x_m() \quad (s > 1).$$

Using the second as a recursion formula, we get

$$L_m^{(s)} / L_m = (L_{m-s+1}^{(1)} / L_{m-s+1})^{p^{(s-1)n}} + f(x_{m-s+2}, \cdots, x_m),$$

where every term of f involves one of the variables displayed, but may involve also $x_1, \cdots, x_{m-s+1}$. Similarly for f in (11) and (12). Hence, for any $s \leqq m - 1$,

$$Q_{ms} = L_{m-s}^{\pi_s} + f(x_{m-s+1}, \cdots, x_m), \qquad \pi_s \equiv p^{(s+1)n} - p^{sn}. \tag{11}$$

A more convenient notation proves to be

$$q_{ms} = Q_{mm-s} = L_s^{P_s} + f(x_{s+1}, \cdots, x_m), \qquad P_s = \pi_{m-s}. \tag{12}$$

Let K be any regular covariant of a system Σ of m-ary forms, *i. e.*, one not divisible by x_m. Let e_{m-1} be the exponent $\geqq 0$ of the highest power of x_{m-1} which divides the sum K_{m-1} of the terms of K not involving x_m. Then

$$K_{m-1} = L_{m-1}^{e_{m-1}}(K_{m-2} + x_{m-1}f),$$

where K_{m-2} is a function of $x_1, \cdots, x_{m-2}$ not identically zero. Let e_{m-2} be the exponent $\geqq 0$ of the highest power of x_{m-2} which divides K_{m-2}. Then

$$K_{m-2} = L_{m-2}^{e_{m-2}}(K_{m-3} + x_{m-2}f') \qquad (K_{m-3} \not\equiv 0).$$

Proceeding similarly, and noting that $x_1 = L_1$, we get

$$K = SL_1^{e_1} L_2^{e_2} \cdots L_{m-1}^{e_{m-1}} + \cdots \qquad (S \not\equiv 0). \tag{13}$$

The coefficient S is called the leader of the regular covariant K.

Denote by $[S, g_1, \cdots, g_{m-1}]$ the set of all the regular covariants (13) in which S is a given leader and

$$e_1 \equiv g_1 \pmod{P_1}, \cdots, e_{m-1} \equiv g_{m-1} \pmod{P_{m-1}},$$

where $g_1, \cdots, g_{m-1}$ are given integers. From this set we may select a finite number of covariants

$$K_i = SL_1^{e_{i1}} L_2^{e_{i2}} \cdots L_{m-1}^{e_{im-1}} + \cdots \qquad (i = 1, \cdots, t),$$

such that, in any covariant (13) of the set,

$$e_1 \geqq e_{i1}, \cdots, e_{m-1} \geqq e_{im-1} \qquad (\text{for a certain } i \leqq t).$$

I have given elsewhere* a simple proof by induction of this lemma on a set of monomials forms in $m-1$ arbitrary variables $L_1, \cdots, L_{m-1}$. For $m = 3$, a direct proof was given in § 3. Thus any covariant of the set has the form

$$K = S\prod_{j=1}^{m-1} L_j^{e_{ij}+r_jP_j} + \cdots \qquad (r_i \text{ integers } \geqq 0)$$

The covariant, of the same order as K,

$$K_i \prod_{j=1}^{m-1} q_{mj}^{r_j}$$

has the same first term as K. Their difference either has the factor x_m, so that

$$K = f(K_1, \cdots, K_\tau, q_{m1}, \cdots, q_{mm-1}) + I \qquad (I \text{ irregular}), \tag{14}$$

*American Journal of Mathematics, October, 1913.

for $\tau = t$, or is of the form

$$S' L_1^{e_1'} L_2^{e_2'} \cdots L_{m-1}^{e'_{m-1}} + \cdots \qquad (S' \neq 0),$$

$$\begin{gathered} e'_{m-1} > e_{m-1}; \quad \text{or} \quad e'_{m-1} = e_{m-1}, \quad e'_{m-2} > e_{m-2}; \quad \cdots; \\ \text{or} \quad e'_{m-1} = e_{m-1}, \quad e'_{m-2} = e_{m-2}, \quad \cdots, \quad e'_3 = e_3, \quad e'_2 > e_2, \end{gathered} \tag{15}$$

as shown by an inspection of the relations preceding (13).

Let $K_1, \cdots, K_\tau$ serve for all the sets as did $K_1, \cdots, K_t$ for the one set considered above. Thus any covariant is of the form (14) or

$$K = f(K_1, \cdots, K_\tau, q_{m1}, \cdots, q_{m\,m-1}) + K',$$

where K' is a regular covariant for which one of the alternatives (15) holds. Since $\Sigma d_i e_i$ is constant where d_i is the total degree of L_i, repetitions of the process lead ultimately to a relation (14). Hence, as at the end of § 3, every covariant is expressible as a polynomial in

$$K_1, \cdots, K_\tau, q_{m1}, \cdots, q_{m\,m-1}, L_m.$$

Degrees and weights of the coefficients of a binary covariant.

5. **Theorem.** *If μ is the index and ω the order of a homogeneous covariant K of a binary form f of order q in the $GF[p^n]$, then $2\mu + \omega$ is divisible by the greatest common divisor d of q and $p^n - 1$. The degrees of the various terms of the various coefficients of K differ by multiples of $(p^n - 1)/d$.*

Let λ be a primitive root in the field. Under the transformation

$$x = \lambda x', \qquad y = \lambda y'$$

of determinant λ^2, let

$$f = a_0 x^q + a_1 x^{q-1} y + \cdots = a'_0 x'^q + \cdots \qquad (a'_i = \lambda^q a_i),$$

$$K = S_0 x^\omega + S_1 x^{\omega-1} y + \cdots = \lambda^\omega (S_0 x'^\omega + S_1 x'^{\omega-1} y' + \cdots).$$

Since K is a covariant of index μ,

$$\sum_{j=0}^{\omega} S_j(a'_i)\, x'^{\omega-j} y'^j = (\lambda^2)^\mu K,$$

$$S_j(a'_i) = \lambda^{2\mu+\omega} S_j(a_i),$$

$$2\mu + \omega \equiv q d_t \pmod{p^n - 1}, \tag{16}$$

where $d_t = e_0 + \cdots + e_q$ is the degree of a term

$$t = \tau a_0^{e_0} \; a_1^{e_1} \; \cdots \; a_q$$

of $S_j(a_i)$. The theorem follows from (16).

Corollary. If $p > 2$, there is no covariant of odd order ω of a binary form of even order q.

6. **Theorem.** *The weight of any term of the leading coefficient S_0 of a covariant K differs from the index μ by a multiple of $p^n - 1$.*

Employing the transformation $x = x'$, $y = \lambda y'$, we get

$$a_i' = \lambda^i a_i, \qquad S_j(a_i') = \lambda^{\mu+j} S_j(a_i),$$

$$w_j = e_1 + 2e_2 + \cdots + q e_q \equiv \mu + j \pmod{p^n - 1}. \tag{17}$$

Fundamental system of covariants of the binary quadratic form modulo 3.

7. The linearly independent invariants (all absolute) of

$$f = ax^2 + 2bxy + cy^2$$

under linear transformation modulo 3 are*

$$1, \quad \Delta = b^2 - ac, \quad I = (a^2 - 1)(b^2 - 1)(c^2 - 1), \quad q = (a + c)(b^2 + ac - 1), \quad \Delta^2. \tag{18}$$

We require the seminvariants, viz., the invariants under

$$x \equiv x' + y', \qquad y \equiv y' \pmod 3. \tag{19}$$

The latter replaces f by f', in which

$$a' \equiv a, \qquad b' \equiv a + b, \qquad c' \equiv a - b + c \pmod 3. \tag{20}$$

Hence the following functions are seminvariants:

$$a, \ a^2, \ a\Delta, \ a\Delta^2, \ a^2\Delta, \ B = (a^2 - 1)b. \tag{21}$$

Any seminvariant is a linear homogeneous function of the eleven linearly independent seminvariants (18) *and* (21). Indeed,† after subtracting constant multiples of these eleven, it remains only to consider a seminvariant

$$S = \alpha_1 bc^2 + \alpha_2 bc + \alpha_3 b + \alpha_4 b^2 c^2 + \alpha_5 b^2 c + \alpha_6 b^2 + \beta c^2 + \gamma c,$$

in which α_1, α_2 are linear functions of a^2, a, 1; and $\alpha_3, \cdots, \alpha_6$ are linear functions of a, 1; while the coefficients of these linear functions and β, γ are constants (independent of a, b, c). In the increment to S under (20), the coefficient of bc^2 is $-a\alpha_4$, whence $\alpha_4 \equiv 0$; then that of $b^2 c$ is $\alpha_1 \equiv 0$; then that of bc is $\beta - a\alpha_5$, whence $\beta \equiv \alpha_5 \equiv 0$; then that of b^2 is $-\alpha_2 \equiv 0$; then that of b is $-\gamma - a\alpha_6$, whence $\gamma \equiv \alpha_6 \equiv 0$. Now $S = \alpha_3 b$, whose increment is $\alpha_3 a$, whence $\alpha_3 \equiv 0$.

* These Transactions, vol. 8 (1907), p. 209. I now write q for Q.

† A shorter, but less elementary, proof may be based upon the classes of forms f under the group generated by (19).

Any polynomial in Δ, I, q, a, B may be expressed as a linear function of the eleven seminvariants (18), (21) by means of the relations

$$I^2 \equiv -I, \quad q^2 \equiv I - \Delta^2 + 1,$$

$$I\Delta \equiv Iq \equiv Ia \equiv IB \equiv q\Delta \equiv qB \equiv aB \equiv 0, \tag{22}$$

$$\Delta B \equiv B, \quad a^2\Delta^2 \equiv \Delta^2 + a^2\Delta - \Delta, \quad aq \equiv a^2\Delta^2 - a^2, \quad B^2 \equiv \Delta(1 - a^2),$$

and the evident relations $a^3 \equiv a, \Delta^3 \equiv \Delta$.

8. By the Corollary in § 5, there is no covariant of odd order. We now determine the quadratic covariants

$$C = Sx^2 + 2S_1 xy + S_2 y^2.$$

Since $C - if$ is a covariant if i is a constant multiple of one of the invariants (18), we may subtract from S a constant multiple of $a, a\Delta, a\Delta^2$ or aq. By (22),

$$aq = \Delta^2 + a^2\Delta - \Delta - a^2.$$

Hence, by § 7, we may assume that

$$S = V + ra^2 + sB \qquad (V \text{ an invariant}). \tag{23}$$

Under the transformation of determinant unity

$$x \equiv y', \qquad y \equiv -x' \pmod 3, \tag{24}$$

f becomes f', in which $a' \equiv c, b' \equiv -b, c' \equiv a$. Hence

$$S_1(c, -b, a) \equiv -S_1(a, b, c), \quad S_2 \equiv S(c, -b, a) \equiv V + rc^2 + sB', \tag{25}$$

$$B' = -(c^2 - 1)b. \tag{26}$$

Under the transformation (19),

$$C = Sx'^2 + 2(S + S_1)x'y' + (S + 2S_1 + S_2)y'^2.$$

Since C is to be a covariant, this must be identical with

$$S'x'^2 + 2S_1'x'y' + S_2'y'^2, \qquad S_i' \equiv S_i(a', b', c'),$$

where $a', \cdots$ are given by (20). Hence

$$S' \equiv S, \qquad S_1' \equiv S + S_1, \qquad S_2' \equiv S + 2S_1 + S_2 \pmod 3. \tag{27}$$

The first condition is satisfied since S is a seminvariant. By the third condition and the invariance of V, we get

$$S_1 \equiv V + r(-\Delta - ab - bc) + sM, \qquad M = (b^2 + ac)(c - a).$$

By (25_1), we get

$$-S_1 \equiv V + r(-\Delta + ab + bc) - sM.$$

Hence, by addition, $V = r\Delta$. Thus

$$S_1 = -rb(a+c) + sM, \qquad M \equiv (b^2+ac)(c-a). \tag{28}$$

Since (27_2) is seen to follow, C is a covariant. For $r \equiv 0$, $s \equiv 1$, we get

$$C_1 = Bx^2 + 2Mxy + B'y^2, \tag{29}$$

a covariant of index unity. Indeed, for $x \equiv -x'$, $y \equiv y'$,

$$C_1(a', b', c', x', y') \equiv -C_1(a, b, c, x', y'), \qquad a' \equiv a,\ b' \equiv -b,\ c' \equiv c.$$

For $r \equiv 1$ $s \equiv 0$, we get the absolute covariant

$$C_2 = (\Delta + a^2)x^2 - 2b(a+c)xy + (\Delta + c^2)y^2. \tag{30}$$

By use of (22), or by the table in § 13, we find that

$$\Delta C_1 \equiv C_1, \quad qC_1 \equiv 0, \quad \Delta C_2 \equiv C_2 + qf, \quad qC_2 \equiv (\Delta^2 - 1)f. \tag{31}$$

9. We readily show that any covariant of order $6t$ is of the form $P + LC$, where C is a covariant of order $6t-4$, P is a polynomial in the covariants (including invariants) already found and Q, where, by (1),

$$L = x^3y - xy^3, \qquad Q = x^6 + x^4y^2 + x^2y^4 + y^6. \tag{32}$$

First, let t be odd. Then the covariants

$$if^{3t},\ iQ^t,\ C_1^{3t},\ C_2^{3t} \qquad (i \text{ an invariant})$$

have as coefficients of x^{6t}

$$ai,\ i,\ B,\ \Delta + a^2,$$

respectively. The linear combinations of the latter give all the seminvariants (18), (21). Next, let t be even. Then

$$f^{3t},\ \Delta f^{3t},\ iQf^{3t-3},\ QC_1^{3t-3},\ i_1Q^t \quad (i = 1, \Delta, \Delta^2;\ i_1 = I, \Delta, \Delta^2, q)$$

have as coefficients of x^{6t}

$$a^2,\ a^2\Delta,\ ai,\ B,\ i_1.$$

10. **Lemma.** *If the order ω of a covariant C of a binary quadratic form modulo 3 is not divisible by 3, its leading coefficient S is a linear homogeneous function of the seminvariants* (18), (21), *other than* $1, I, q$.

Under the transformation (19),

$$C = Sx^\omega + S_1x^{\omega-1}y + \cdots \equiv Sx'^\omega + (S_1 + \omega S)x'^{\omega-1}y' + \cdots.$$

For a covariant C, the final sum equals

$$Sx'^\omega + S_1'x'^{\omega-1}y' + \cdots, \qquad S_1' \equiv S_1(a', b', c'),$$

where $a', \cdots$ are given by (20). Hence

$$S_1' - S_1 \equiv \omega S \pmod 3. \tag{33}$$

Set

$$S_1 = ka^2 b^2 c^2 + t \qquad (t \text{ of degree} < 6).$$

Applying (20), we get

$$S_1' = ka^2 (a+b)^2 (a-b+c)^2 + t' \equiv ka^2 (ar + b^2 + bc + b^2 c^2) + t',$$

where r is of degree 3 and t' of degree < 6. Hence, by (33),

$$\omega S \equiv k (ar + a^2 b^2 + a^2 bc) + t' - t \pmod 3.$$

Since ω is prime to 3, S is of degree < 6. Hence S does not contain the term $a^2 b^2 c^2$, which occurs in I but not in any other seminvariant (18), (21).

Next, if $S = 1 + \sigma$, where σ is a function of a, b, c without a constant term, IC is a covariant C' with $S' = I$.

Finally, let $S = q + \alpha_1 + \alpha_2 \Delta + \alpha_3 \Delta^2 + tB$, where t is a constant and the α_i are functions of a. By (22),

$$qS = I - \Delta^2 + 1 + \alpha_1 q,$$

which has the term $a^2 b^2 c^2$ (from I).

11. Covariants C of order $\omega = 6t + 2$. For t odd, the covariants

$$f^{3t+1},\ Q^t f,\ C_2^{3t+1},\ f^{3t} C_2,\ C_1^{3t} C_2$$

have as coefficients of x^ω

$$a^2,\ a,\ \Delta^2 - a^2 \Delta + a^2,\ a\Delta + a,\ B,$$

respectively. Linear combinations of products of these by invariants give the seminvariants (21) and Δ^2, Δ. Hence, by the Lemma, $C = P + LC'$, where P is a polynomial in the known covariants. For t even,

$$fQ^t,\ f^4 Q^{t-1},\ C_2 Q^t,\ C_1 Q^t$$

have a, a^2, $\Delta + a^2$, B as coefficients of x^ω.

12. Covariants C of order $\omega = 6t + 4$. The function

$$f_4 = ax^4 + bx^3 y + bxy^3 + cy^4, \tag{34}$$

which is congruent modulo 3 to the quadratic form f for all integral values of x, y, is readily seen to be an absolute covariant (also when a, b, c are arbitrary variables).

The coefficients of x^ω in the covariants

$$f_4 Q^t,\ f^2 Q^t,\ C_1 C_2 Q^t,\ C_1^2 Q^t$$

are a, a^2, B, $\Delta - a^2 \Delta$. Linear combinations of their products by invariants give all seminvariants not containing $1, I, q$.

13. We have now completed the proof of the

Theorem. *As a fundamental system of rational integral covariants of the binary quadratic form modulo* 3 *we may take*

$$\Delta,\ q,\ f,\ C_1,\ C_2,\ f_4,\ L,\ Q. \tag{35}$$

No one of these eight concomitants is a rational integral function of the remaining seven. To prove this, consider their expressions for five special sets of values of a, b, c (in fact, those giving the non-equivalent f's under the group of transformations of determinant unity):

Case	f	Δ	q	C_1	C_2	f_4
1	0	0	0	0	0	0
2	x^2	0	-1	0	x^2	x^4
3	$-x^2$	0	1	0	x^2	$-x^4$
4	x^2+y^2	-1	0	0	0	x^4+y^4
5	$2xy$	1	0	$-x^2+y^2$	x^2+y^2	x^3y+xy^3

To show that L and Q are not functions of the remaining concomitants, we use case 1. For f_4, use case 4. No linear relation holds between f, C_1, C_2 in which C_1 is present, since C_1 is of index 1, while f and C_2 are absolute covariants. Now $f \neq kC_2$ by case 4; $C_2 \neq kf$ by case 5. Next, $q \neq F(\Delta)$ by 2 and 3; $\Delta \neq F(q)$ by 4 and 5.

We note the syzygies

$$fC_1 = 2(\Delta^2+\Delta)L, \qquad fC_2 = (1+\Delta)f_4,$$
$$C_2^2 - C_1^2 = (\Delta+1)^2 f^2, \qquad C_2^3 - ff_4 = \Delta Q.$$

In the algebraic theory, Δ and f form a fundamental system.

THE UNIVERSITY OF CHICAGO,
January, 1913.

ON TRANSCENDENTALLY TRANSCENDENTAL FUNCTIONS*

BY

R. D. CARMICHAEL

1. Introduction.—Following the classification made by E. H. MOORE in his fundamental paper† on transcendentally transcendental functions, we shall say that a transcendental function y of x is algebraically or transcendentally transcendental according as it does or does not satisfy a differential equation of the form

$$H(x, y, y^{(1)}, \cdots, y^{(k)}) = 0, \qquad y^{(i)} = \frac{d^i y}{dx^i} \quad (i = 1, \cdots, k),$$

where H is a polynomial in $x, y, y^{(1)}, \cdots, y^{(k)}$ with constant coefficients. The existence of transcendentally transcendental functions was first shown by HÖLDER,‡ who proved that the gamma function belongs to this class of functions. HURWITZ§ obtained important results concerning these functions and pointed out a special example. Others who have written on the subject are GRONWALL,‖ BARNES,¶ TIETZE,** MORDOUCHAY-BOLTOVSKY,†† and STRIDSBERG.‡‡

The principal result of this paper is stated in section 2 as a general theorem; it is in the nature of an extension of a result due to HURWITZ. It furnishes an immediate means of writing out an unlimited number of transcendentally transcendental functions. In section 3, by means of number-theoretic considerations taken in connection with preceding results, I prove the transcendentally transcendental character of the class of functions defined in equation (5) below; and in section 4 special cases of these functions are given which satisfy functional equations of simple type.

* Presented to the Society (Chicago) December 30, 1911.

† Mathematische Annalen, vol. 48 (1896), pages 49–74.

‡ Mathematische Annalen, vol. 28 (1887), pages 1–13.

§ Annales de l'école normale supérieure, ser. 3, vol. 6 (1889), pages 327–332.

‖ Öfersigt K. Svenska Vetenskap. Akad. Förh., 1898.

¶ Proceedings of the London Mathematical Society, ser. 2, vol. 2 (1904), pages 280–292. Seereferences in this paper.

** Monatshefte für Mathematik und Physik, vol. 16 (1905), pages 329–364. See the introduction to this paper for further references.

†† Matematischiski Sbornik, vol. 27 (1909), pages 360–406.

‡‡ Arkiv för Mat., Astr. och Fys., vol. 6 (1910), numbers 15 and 18.

2. General Theorem.—Let us consider the function of x defined by the convergent infinite power series*

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \tag{1}$$

where $a_0, a_1, a_2, \cdots$ are rational fractions in their lowest terms. We seek properties of the coefficients a_n which are necessary if y is to satisfy an algebraic differential equation, that is, an equation of the form

$$P(x, y, y^{(1)}, \cdots, y^{(k)}) = 0, \tag{2}$$

where P is a polynomial in $x, y, y^{(1)}, \cdots, y^{(k)}$. It is clear that if y is an integral of any such differential equation, it is an integral of one of the same form in which the coefficients are restricted to be integers. Accordingly, we suppose that (2) has this property. Let us assume further, as we may without loss of generality, that y satisfies no such equation of the form (2) and of order lower than k, and that the degree of (2) in $y^{(k)}$ is equal to or less than the degree in $y^{(k)}$ of any other such equation of order k which y satisfies.

If we differentiate (2) with respect to x an appropriate number of times, and in the result put $x = 0$ and then clear of fractions, we have finally (see HURWITZ, l. c., p. 329, equation (8)) an equation of the form

$$y_0^{(n)}(b_0 + b_1 n + \cdots + b_\gamma n^\gamma) = G_n(y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}) \tag{3}$$

where $b_0, b_1, \cdots, b_\gamma, \gamma$ are integers independent of n, and G_n is a polynomial in $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ with integral coefficients; the equation being valid and sufficing for the determination of $y_0^{(n)}$ for every value of n greater than some particular integer N. Here $y_0, y_0^{(1)}, \cdots, y_0^{(n)}$ are what $y, y^{(1)}, \cdots, y^{(n)}$ become when $x = 0$.

An upper limit to the degree of G_n in its arguments is readily found. Suppose that equation (2) is of degree t in $y, y^{(1)}, \cdots, y^{(k)}$. Then, after any number of differentiations, the resulting equation is of degree t (at most) in y and its derivatives. If in this we put $x = 0$, the equation so obtained is at most of degree t in $y_0, y_0^{(1)}, y_0^{(2)}, \cdots$. Hence the degree of G_n in its arguments $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ is at most t.

Now let m be any integer such that, for all values of n equal to or greater than m, equation (3) is valid and the polynomial

$$h(n) = b_0 + b_1 n + \cdots + b_\gamma n^\gamma$$

is different from zero. Let D be the least common multiple of the denominators of $y_0, y_0^{(1)}, \cdots, y_0^{(m-1)}$. Then form the value of $y_0^{(m+s)}$, $s \gtreqless 0$, by

* By an "infinite power series" I mean a power series with an infinite number of coefficients different from zero. A zero coefficient in (1) we shall suppose to be written in the form $0/1$.

means of a successive use of (3); we shall show by induction that the result may be written in the form

$$(4)\quad y_0^{(m+s)} = \frac{\bar{A}_{m+s}}{D^{s+1}[h(m)]^s[h(m+1)]^{s-1}\cdots[h(m+s-1)]^1 h(m+s)},$$

where $\bar{A}_{m+s}$ is an integer. For $s = 0$, (4) takes the form

$$y_0^{(m)} = \frac{\bar{A}_m}{D^1 h(m)}.$$

That this relation is true follows from (3) by putting $n = m$. Consequently the proof of (4) may be completed by showing that if (4) is true for all values of s equal to or less than a given value σ, it is also true for $s = \sigma + 1$. From (3) we have

$$y_0^{(m+\sigma+1)} = \frac{1}{h(m+\sigma+1)} G_{m+\sigma+1}(y_0, y_0^{(1)}, \cdots, y_0^{(m+\sigma)}),$$

where $G_{m+\sigma+1}$ is a polynomial of degree t (at most) in its arguments. The rational numbers $y_0, y_0^{(1)}, \cdots, y_0^{(m-1)}$ have a common denominator D. Hence if we assume (4) for $s = 0, 1, 2, \cdots, \sigma$, we see that the rational numbers $y_0, y_0^{(1)}, \cdots, y_0^{(m+\sigma)}$ have as a common denominator the denominator in the second member of (4) for $s = \sigma$. Writing each of these numbers as a fraction having this common denominator, and substituting in the last equation above (remembering that G is of degree t at most in its arguments), we are led at once to an equation of the form (4) in which s is replaced by $\sigma + 1$. Hence the proof of (4) is complete.

Now if we put

$$g(x) = (m-1)!\, D^t x(b_0 + b_1 x + \cdots + b_\gamma x^\gamma) \equiv c_0 + c_1 x + \cdots + c_\alpha x^\alpha$$

and remember that $a_n = y_0^{(n)}/n!$, we see that (4) leads at once to the following theorem:*

THEOREM: *If the function y of x defined by the convergent series*

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are rational fractions (which we take to be in their lowest terms), is an integral of an algebraic differential equation whose coefficients are rational functions of x, then there exist positive integers m and t, and a polynomial

$$g(x) = c_0 + c_1 x + \cdots + c_\alpha x^\alpha$$

whose coefficients $c_0, c_1, \cdots, c_\alpha$ are integers independent of x, such that a_n,

* This theorem is in the nature of an extension of the result stated by HURWITZ in the footnote on p. 330 of the paper already referred to, the extension consisting essentially in the determination of Hurwitz's constants $\alpha_0, \alpha_1, \cdots$ and $\beta_0, \beta_1, \cdots$.

$n > m$, has the value

$$a_n = \frac{A_n}{[g(m)]^{t^{n-m}}[g(m+1)]^{t^{n-m-1}} \cdots [g(n-1)]^t g(n)},$$

where A_n is an integer and no one of the numbers $g(m)$, $g(m+1)$, $\cdots$ is zero.

The integer t is the degree in $y, y^{(1)}, \cdots, y^{(k)}$ of the polynomial P in equation (2) defined as in the first paragraph of this section.

COROLLARY I: *For such a function y there exists a positive integer τ such that the denominator [numerator] of a_n, for $n > 1$, is in absolute value less than n^{τ^n}.*

The part of this corollary which refers to the denominators is an immediate consequence of the theorem and the fact that

$$\lim_{n=\infty} n^{-\tau^n}[g(m)]^{t^{n-m}} \cdots [g(n-1)]^t g(n) = 0$$

when $\tau > t$. The part referring to the numerators follows from the other part and the fact that the series for y is convergent for some x different from zero. (It may be necessary to take the value of τ greater in the latter case than in the former.)

COROLLARY II: *If $b_0, b_1, b_2, \cdots$ is any set of rational fractions in their lowest terms, an infinite number of which are different from zero, and if there exists an integer τ such that the numerator of b_n, for increasing n, is ultimately less than n^{τ^n}, then each of the following series is permanently convergent and represents a transcendentally transcendental entire function:*

$$b_0 + b_1 x + \frac{b_2}{16} x^2 + \cdots + \frac{b_n}{n^{n^n}} x^n + \cdots,$$

$$b_0 + b_1 x + \frac{b_2}{24} x^2 + \cdots + \frac{b_n}{(n^n)!} x^n + \cdots,$$

$$b_0 + \frac{b_1}{a} x + \frac{b_2}{a^{a^2}} x^2 + \frac{b_3}{a^{a^{a^3}}} x^3 + \cdots.$$

In the last example, a is a positive integer equal to or greater than 2, and the denominator in the $(n+1)$th term is $a^{a^{a^{\cdots}}}$, a being repeated n times.

The proof follows at once from the fact that $n^{(t+\tau)^n}$ (t = any positive integer), which, for increasing n, is ultimately greater than n^{t^n} times the numerator of b_n, is at the same time ultimately less than n^{n^n} or $(n^n)!$ or $a^{a^{\cdots}}$. For when the coefficients of the powers of x in any one of the above series are reduced to their lowest terms their denominators fail to satisfy the condition of Corollary I, which is necessary if the function is an integral of an algebraic differential equation.

From the method of proof of this corollary it follows that any of these series will continue to represent a transcendentally transcendental function if any of its denominators are increased (in absolute value) in any way, provided that the resulting number is still rational. Various other forms of examples suggest themselves at once.

From Corollary I it follows readily that for the numbers $b_0, b_1, \cdots$ in Corollary II we may take any set of rational numbers (an infinite number of which are different from zero) such that $y = b_0 + b_1 x + b_2 x^2 + \cdots$ is an integral of an algebraic differential equation.

3. Examples of transcendentally transcendental functions.—In Corollary II above, and the remarks following it, we have suggested an unlimited number of examples of transcendentally transcendental functions. In the present section we obtain others of a different character.

From equation (3) the following result is readily deduced: *If y is defined as in the theorem above, then there exists an integer m and a polynomial $h(x) = b_0 + b_1 x + \cdots + b_\nu x^\nu$ whose coefficients are integers independent of x, such that every divisor d of the denominator of a_n which is prime to $n!$ and to the denominator of every a_i, $i < n$, is a factor of $h(n)$.* This theorem is practically proved by Hurwitz (l. c., pp. 329, 330) though not explicitly stated. We have the following immediate consequence of this result:

LEMMA. *If y (as defined in the theorem above) is an integral of an algebraic differential equation, and if the denominator of a_n is written in the form $r_n s_n$ where s_n is prime to $n!$ and to the denominator of every a_i, $i < n$, then there exists an integer ϵ such that, for increasing n, s_n ultimately becomes and remains less than n^ϵ.*

This lemma we shall now employ to prove the following:

Let $b_0, b_1, b_2, \cdots$ be a set of rational fractions in their lowest terms, an infinite number of which are different from zero. Let them have the property that a number N exists such that, for $n > N$, neither the numerator nor the denominator of b_n (when $b_n \neq 0$) contains a prime factor greater than n. Let α and β be any relatively prime integers one of which is in absolute value greater than 1. Then the series

$$(5)\quad y = b_0 + \frac{b_1}{(\alpha-\beta)^{k_{11}}} x + \cdots + \frac{b_n}{(\alpha-\beta)^{k_{1n}}(\alpha^2-\beta^2)^{k_{2n}}\cdots(\alpha^n-\beta^n)^{k_{nn}}} x^n + \cdots,$$

if it converges, defines a function which is transcendentally transcendental. Here the numbers k_{ij} are integers (positive, negative, or zero) and k_{nn} is equal to or greater than 1 for an infinite number of values of n for which $b_n \neq 0$.

Let $n_1, n_2, n_3, \cdots$ be an infinite sequence of numbers greater than N and such that $k_{n_i n_i} \geqq 1$ while $b_{n_i} \neq 0$. Denote by β_n the coefficient of x^n in series (5), when reduced to its lowest terms, and let s_{n_i} be the greatest factor

of the denominator of β_{n_i} which is prime to $n_i!$ and to the denominator of every b_k, $k < n_i$. We shall show first that s_{n_i}, for every i, contains the greatest factor F_i of $\alpha^{n_i} - \beta^{n_i}$ which is prime to every $\alpha^s - \beta^s$, $s < n_i$. It is well known* that every prime factor q of $\alpha^{n_i} - \beta^{n_i}$ which is not contained in any $\alpha^s - \beta^s$, $s < n_i$, is of the form $\lambda n_i + 1$. It is therefore greater than any factor of either the numerator or the denominator of any b_k, $k \lesseqgtr n_i$, and hence it is not contained in any of these. Similarly, q is not contained in $n_i!$. From these results and the fact that q is prime to every $\alpha^s - \beta^s$, $s < n_i$, it follows that F_i is a factor of s_{n_i}.

In view of our lemma, the proof of the transcendentally transcendental character of y as defined in (5) will be completed if we show that for every integer ϵ we have

$$\lim_{n_i=\infty} \frac{n_i^\epsilon}{F_i} = 0.$$

To effect the proof of this we shall need certain properties of the number F_i, and these we shall now obtain.

Let $Q_n(x) = 0$ be the equation whose roots are the primitive nth roots of unity without repetition, the coefficient of the leading term in $Q_n(x)$ being unity. The degree of $Q_n(x)$ is $\varphi(n)$, where φ denotes Euler's φ-function. Form the quantity $Q_n(\alpha, \beta)$ defined by the equation

$$Q_n(\alpha, \beta) = \beta^{\phi(n)} Q_n\left(\frac{\alpha}{\beta}\right).$$

It is evident that $Q_n(\alpha, \beta)$ is a homogeneous form of degree $\varphi(n)$ in α, β. I have shown† that the greatest common divisor of $Q_n(\alpha, \beta)$ and $\alpha^{n/p} - \beta^{n/p}$ is 1 or p, p being any prime factor of n. From this it follows readily that $Q_n(\alpha, \beta)$ has no factor in common with the numbers $\alpha^s - \beta^s$, $s < n$, except possibly prime factors of n and their products. Hence it follows that

$$F_i \leqq \frac{|Q_{n_i}(\alpha, \beta)|}{n_i}.$$

Now suppose that ρ lies between the two quantities $|\alpha|$ and $|\beta|$ and is greater than 1. By using the product form for $Q_n(\alpha, \beta)$ obtained from the product form‡ for $Q_n(x)$, one may show without difficulty that we have

$$\lim_{n_i=\infty} \frac{\rho^{\phi(n_i)}}{|Q_{n_i}(\alpha, \beta)|} = \lim_{n_i=\infty} \frac{\Pi(\alpha_1^{n/p} - \beta_1^{n/p}) \cdot \Pi(\alpha_1^{n/pqr} - \beta_1^{n/pqr}) \cdots}{(\alpha_1^n - \beta_1^n)\,\Pi(\alpha_1^{n/pq} - \beta_1^{n/pq}) \cdots} = 0,$$

* Lucas, American Journal of Mathematics, vol. 1 (1878), p. 291.
† American Mathematical Monthly, vol. 16 (1909), pages 153–159.
‡ See Bachmann's *Kreisteilung*, p. 16.

where

$$\alpha_1 = \frac{\alpha}{\rho}, \qquad \beta_1 = \frac{\beta}{\rho},$$

and $p, q, r, \cdots$ are the different prime factors of n. Hence one sees readily that the proof of our theorem will be completed if we show that

$$\lim_{n_i=\infty} \frac{n_i^{\epsilon+1}}{\rho^{\phi(n_i)}} = 0$$

for every integer ϵ; and the existence and value of the last limit follow at once from the obvious fact that for increasing n_i

$$\rho^{\phi(n_i)} \quad \text{ultimately becomes and remains greater than} \quad n_i^{\epsilon+2}.$$

Thus we have completed the demonstration of the transcendentally transcendental character of y as defined in equation (5).

4. Transcendentally transcendental functions satisfying functional equations.—In the preceding sections we have established the transcendentally transcendental character of functions which may be written down in unlimited number. Usually, however, there will be nothing to ensure that a function so defined shall possess valuable or interesting special properties. In the present section we give several special cases of functions of the general form in equation (5) so chosen that they satisfy functional equations of simple type and of such form as to make certain that the functions are essentially simple in character and possess a set of important special properties which are easily found. In no case, however, will the detailed discussion of the properties of the functions be given.

Throughout this section we shall denote by α and β two relatively prime integers one at least of which is in absolute value different from unity; and we shall write $\alpha / \beta = q$.

EXAMPLE 1. If $b_1 x + b_2 x^2 + b_3 x^3 + \cdots$ is a convergent infinite power series whose coefficients are rational fractions in their lowest terms such that for every n greater than some preassigned N neither the numerator nor the denominator of b_n (when $b_n \neq 0$) contains a prime factor greater than n, then the function

$$g(x) = \frac{b_1 \beta}{\alpha - \beta} x + \frac{b_2 \beta^2}{\alpha^2 - \beta^2} x^2 + \frac{b_3 \beta^3}{\alpha^3 - \beta^3} x^3 + \cdots$$

is transcendentally transcendental and satisfies the functional equation

$$g(qx) - g(x) = b_1 x + b_2 x^2 + b_3 x^3 + \cdots.$$

Special cases are afforded by taking as the second member of this equation

the developments of the functions

$$\frac{x}{1-x},\quad e^x-1,\quad \log(1+x),\quad \sin x,\quad \cos x-1,\quad \cdots.$$

In particular we have a transcendentally transcendental function

$$g(x)=\frac{x}{a-1}+\frac{x^2}{a^2-1}+\frac{x^3}{a^3-1}+\cdots,\quad g(ax)-g(x)=\frac{x}{1-x},$$

a being any integer greater than 1.

EXAMPLE 2:

$$g(x)=1+\frac{x}{(q-1)^n}+\cdots+\frac{x^m}{(q-1)^n(q^2-1)^n\cdots(q^m-1)^n}+\cdots;$$

$$g(q^n x)-ng(q^{n-1}x)+\frac{n(n-1)}{2!}g(q^{n-2}x)-\cdots+(-1)^n g(x)=xg(x).$$

For the special case when $n=1$ we have

$$g(x)=1+\frac{x}{(q-1)}+\frac{x^2}{(q-1)(q^2-1)^2}+\cdots;\quad g(qx)=(1+x)g(x).$$

The transcendentally transcendental character of the last function has been previously established by Stridsberg (see the first paper referred to above).

EXAMPLE 3:

$$g(x)=\frac{x}{\alpha-\beta}+\frac{\gamma x^n}{(\alpha-\beta)(\alpha^n-\beta^n)}+\frac{\gamma^{1+n}x^{n^2}}{(\alpha-\beta)(\alpha^n-\beta^n)(\alpha^{n^2}-\beta^{n^2})}$$
$$+\frac{\gamma^{1+n+n^2}x^{n^3}}{(\alpha-\beta)(\alpha^n-\beta^n)\cdots(\alpha^{n^3}-\beta^{n^3})}+\cdots,$$
$$g(\alpha x)-g(\beta x)-g(\gamma x^n)=x,$$

where γ is any rational number and n is an integer greater than 1.

EXAMPLE 4:

$$g(x)=1+\frac{x}{\alpha-\beta}+\frac{x^n}{\alpha^n-\beta^n}+\frac{x^{n^2}}{\alpha^{n^2}-\beta^{n^2}}+\frac{x^{n^3}}{\alpha^{n^3}-\beta^{n^3}}+\cdots,$$
$$g(\alpha x)-g(\beta x)-g(\alpha x^n)+g(\beta x^n)=x,$$

where n is an integer greater than 1.

EXAMPLE 5:

$$g(x)=1+\frac{x}{\alpha-\beta}+\frac{x^2}{(\alpha-\beta)(\alpha^2-\beta^2)}+\frac{x^3}{(\alpha-\beta)(\alpha^2-\beta^2)(\alpha^3-\beta^3)}+\cdots,$$
$$g(\alpha x)-g(\beta x)=xg(x).$$

EXAMPLE 6: The functional equation

$$g(\alpha x) - g(\beta x) - g(x^2) - g(x^3) = x$$

possesses a unique solution which is analytic at the point zero. This function is transcendentally transcendental.

It is obvious that such special examples might be multiplied indefinitely.

INDIANA UNIVERSITY.

SUR LES CLASSES V NORMALES*

PAR

MAURICE FRÉCHET

Considérons une classe (Δ) d'éléments de nature quelconque mais satisfaisant aux conditions suivantes:†

1° C'est une classe (L). 2° Elle est parfaite. 3° Elle est compacte. 4° Si tous les éléments limites d'une suite infinie d'éléments de la classe (suite qui a nécessairement au moins un élément limite d'après 3°) coïncident, cette suite est convergente. 5° A tout élément A de (Δ) correspond une suite infinie dénombrable d'ensembles fermés $Q_i(A)$ possédant les propriétés suivantes: *a*) $Q_{i+1}(A)$ appartient à $Q_i(A)$; *b*) à tout entier m correspond un entier n tel que tout Q_n à qui appartient un élément B appartient à $Q_m(B)$ quel que soit B; *c*) A est *intérieur* à tous les $Q_i(A)$; *d*) les ensembles $Q_1(A), Q_2(A), \cdots, Q_n(A), \cdots$ ont un seul élément commun, à savoir A.

Les théorèmes que vous établissez à partir du Théorème 8 sont démontrés pour une classe satisfaisant aux conditions précédentes énoncées, soit explicitement, soit implicitement comme pour 4° dans la démonstration du Théorème 7, ou pour 2° par suite de la légère différence dans la définition du mot "intérieur" que vous me signalez. Votre classe (D) est soumise en outre à la condition—*dont je n'aurai pas besoin*—que tout ensemble dérivé est fermé et de plus votre "enclosable property" est *un peu* plus restrictive que la condition 5° ci-dessus.

Ceci étant, je vais démontrer d'abord que: *toute classe* (Δ) *est une classe V normale. Il en résultera que les théorèmes que vous démontrez pour la classe* (Δ) *peuvent être considérés comme déjà démontrés dans ma Thèse.* (Sur quelques points du Calcul Fonctionnel, Rendiconti del Circolo Matematico di Palermo, 1906, t. 22, pp. 1–74).‡

Par contre les théorèmes où vous ne faites pas usage de l' "enclosable property" *restent essentiellement nouveaux et constituent une importante généralisation.*

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* Extract from a letter to Professor E. R. HEDRICK concerning his paper: *On Properties of a Domain for which any Derived Set is Closed*, these Transactions, vol. 12 (1911), p. 285. Presented to the Society, April 26, 1913.

† J'adopte votre terminologie en ajoutant seulement au sens de l'expression: A intérieur à E, la condition que A appartient à E.

‡ Je dois d'ailleurs à l'obligeance de M. T. H. HILDEBRANDT (Ann Arbor) la remarque que la démonstration du théorème de BOREL que j'ai donnée dans ma Thèse page 26 pour une classe V normale s'étend immédiatement à une classe V parfaite si l'on tient compte des résultats de mon second mémoire des Rendiconti de Palerme (Les ensembles abstraits et le Calcul Fonctionnel, 1910, t. 30, pp. 1–26).

M. SIGISMOND JANIZSEWSKI m'avait exprimé verbalement l'opinion très juste qu'il y aurait avantage à ne pas faire intervenir dans la définition de la limite la notion de nombre irrationnel qui entre dans ma définition du voisinage. J'attribue à votre notion d'ensemble "enclosable" un grand intérêt du fait qu'elle répond en effet à ce desideratum d'une manière naturelle. (On pourrait d'ailleurs y arriver simplement aussi en remplaçant le voisinage par une fonction convenable du voisinage, fonction qui ne prendrait que des valeurs inverses de valeurs entières.)

Pour démontrer la proposition annoncée je ferai d'abord deux remarques.

1° En remplaçant au besoin les $Q_i(A)$ convenablement je pourrais supposer que, quel que soit A, $Q_1(A)$ soit formé de la classe (Δ) toute entière.

2° Etant donnés deux éléments distincts A, B, il existe un entier β tel que A appartienne à $Q_1(B)$, $Q_2(B)$, $\cdots$, $Q_\beta(B)$ et n'appartienne pas à $Q_{\beta+1}(B)$. En effet A appartient au moins à $Q_1(B) \equiv \Delta$; s'il appartient à $Q_i(B)$, il appartient à $Q_{i-1}(B)$; et enfin il n'appartient pas à tous les $Q(B)$ sans quoi il coïnciderait avec leur seul élément commun B. De même il existe un nombre α tel que B appartienne à $Q_1(A), \cdots, Q_\alpha(A)$ et non à $Q_{\alpha+1}(A)$.

Définition du voisinage.—J'appellerai maintenant voisinage de deux éléments A, B et je désignerai par la notation (A, B) un nombre égal à zéro si A, B coïncident et à l'inverse du plus petit des deux nombres α, β définis précédemment si A, B sont distincts. On voit que A appartiendra aux $Q(B)$ et B aux $Q(A)$ au moins jusqu'au rang $1/(A, B)$.

Il faut d'abord montrer I. qu'une telle définition satisfait bien aux conditions imposées au voisinage dans ma Thèse, II. qu'elle fournit les mêmes suites convergentes et les mêmes éléments limites que ceux qui sont donnés par la définition de Δ.

I. En ce qui concerne le premier point, il suffit de montrer qu'on peut former une fonction $f(\epsilon) > 0$ et tendant vers zéro avec ϵ, telle que les inégalités

$$(A, C) \leq \epsilon, \qquad (B, C) \leq \epsilon \tag{1}$$

entraînent $(A, B) \leq f(\epsilon)$. Comme d'après sa définition le voisinage est ici toujours ≤ 1, je pourrais prendre $f(\epsilon) = 1$ pour $\epsilon \geq 1$ et ne me préoccuper que des valeurs de $\epsilon < 1$. Je ferai encore quelques remarques préliminaires.

3° Si m et n sont deux nombres satisfaisant à la propriété b) de la condition 5°, il en sera de même de tout couple m', n' où $m' \leq m$ et $n' \geq n$, d'après la propriété a).

4° Si au lieu de se donner m, on se donne un entier quelconque n, il y a un entier correspondant bien déterminé $\phi(n)$ tel que n et m jouissent de la propriété b) pour $m \leq \phi(n)$ et n'en jouissent plus pour $m > \phi(n)$. En effet, on peut prendre au moins $m = 1$; si m_0 est admissible pour m, alors $1, 2, \cdots, m_0 - 1$ le sont aussi; enfin si à un entier n correspondaient pour m toutes les

valeurs entières, tout Q_n à qui appartiendrait un élément B appartiendrait à $Q_1(B), Q_2(B), \cdots$ et par suite se réduirait à leur seul élément commun B. En particulier $Q_n(B)$ se réduirait à B ce qui est impossible (Δ étant parfait), si l'on tient compte de la condition c).

5° D'après la remarque 3° on voit que $\phi(n+1) \geq \phi(n)$. La fonction $\phi(n)$ ne peut d'ailleurs être bornée. Car si elle restait inférieure à l'entier μ, à celui-ci, considéré comme valeur de m, correspondrait, d'après b), une valeur finie ν de n et on aurait $\phi(\nu) \geq \mu$. Ainsi $\phi(n)$ est une fonction monotone de l'entier n, fonction qui tend vers l'infini avec n.

Revenons maintenant aux inégalités (1). ϵ étant supposé plus petit que 1, à chaque valeur de $\epsilon > 0$ correspond un entier N tel que $1/(N+1) < \epsilon \leq 1/N$. On aura donc $(A, C) \leq 1/N$, c'est à dire, que C appartiendra aux $Q(A)$ au moins jusqu'au rang N. Si donc $i \leq N$, C appartiendra à $Q_i(A)$, donc $Q_i(A)$ appartiendra à $Q_{\phi(i)}(C)$. Mais si l'on suppose également $\phi(i) \leq N$, B qui appartient comme les A aux $Q(C)$ jusqu'au rang N, appartiendra à $Q_{\phi(i)}(C)$; donc $Q_{\phi(i)}(C)$ appartiendra à $Q_{\phi(\phi(i))}(B)$. En définitive, si $i \leq N$ et $\phi(i) \leq N$, A appartiendra à $Q_{\phi(\phi(i))}(B)$: le nombre β défini au 2° sera au moins égal à $\phi(\phi(i))$; il en sera de même de α; donc $(A, B) \leq 1/\phi(\phi(i))$.

Si maintenant $N \geq \phi(1)$, il y a au moins un entier i tel qu'on ait à la fois $i \leq N$, $\phi(i) \leq N$. Le plus grand de ces nombres i est un nombre $\psi(N)$ bien déterminé. La fonction $\psi(N)$ est monotone et tend vers l'infini avec N, comme $\phi(N)$. Et d'après ce qui précède, on a $(A, B) \leq 1/\phi(\phi(\psi(N)))$. Enfin N est une fonction déterminée de ϵ, $N(\epsilon)$, qui est monotone et croît indéfiniment quand ϵ tend vers zéro. Si donc on pose

$$f(\epsilon) = \frac{1}{\phi(\phi(\psi(N(\epsilon))))}, \qquad \text{pour } \epsilon \leq \frac{1}{\phi(1)},$$

et

$$f(\epsilon) = 1, \qquad \text{pour } \epsilon > \frac{1}{\phi(1)},$$

on voit qu'on aura, comme conséquence de (1),

$$(A, B) \leq f(\epsilon),$$

$f(\epsilon)$ étant une fonction > 0 qui tend vers zéro avec $\epsilon > 0$ *et qui est indépendante de* A, B, C.

II. Jusqu'ici la notion d'éléments limites dans Δ n'est même pas intervenue. Supposons que les suites convergentes dans Δ et leurs limites soient définies de façon à satisfaire aux conditions imposées à Δ. Je vais montrer que si $A_1, A_2, \cdots, A_p, \cdots$ est une suite convergente vers B, le voisinage (B, A_n) tend vers zéro et réciproquement.

Si la suite des A_p converge vers B, comme B est *intérieur* à $Q_n(B)$, les A_p doivent appartenir à $Q_n(B)$ à partir d'un certain rang p_n. Mais alors $Q_n(B)$ appartient à $Q_{\phi(n)}(A_p)$ pour $p > p_n$. Donc pour $p > p_n$, les nombres

α, β définis au n° 2° et relatifs ici à A_p et B seront respectivement au moins égaux à $\phi(n)$ et n; ils seront donc au moins égaux au plus petit de ces deux-ci que nous pourrons désigner par $\theta(n)$. Donc $(A_p, B) \leq 1/\theta(n)$ pour $p > p_n$. Comme $\theta(n)$ tend vers l'infini avec n, ceci prouve que (A_p, B) tend vers zéro quand p croit indéfiniment.

Réciproquement supposons que des éléments $B, A_1, A_2, \cdots$ soient tels que (B, A_p) tende vers zéro avec $1/p$. Cela veut dire qu'à tout entier r on peut faire correspondre un entier q tel que

$$(A_q, B) < \frac{1}{r}, \qquad (A_{q+1}, B) < \frac{1}{r}, \qquad \cdots;$$

autrement dit $A_q, A_{q+1}, \cdots$ appartiendront aux ensembles $Q_1(B), Q_2(B), \cdots, Q_r(B)$. Tous les éléments limites de la suite $A_1, A_2, \cdots$ appartiendront donc aussi à ces ensembles *qui sont fermés*. Comme r est arbitraire, ces éléments limites coïncideront avec le seul élément commun à $Q_1(B), Q_2(B), \cdots, Q_r(B), \cdots$ à savoir B. D'après la condition 4° imposée à Δ, la suite $A_1, A_2, \cdots$ est donc convergente et tend vers B. Ainsi *la classe Δ est bien une classe* (V) (parfaite et compacte). Reste à démontrer qu'elle est *normale*. Or il en est ainsi pour *toute classe* (V) parfaite et compacte. En effet, il n'y a aucune difficulté à prouver qu'une telle classe admet une généralisation du théorème de Cauchy; et d'autre part le fait qu'elle est séparable résulte des théorèmes n^{os} 4 et 5 de mon second mémoire* ou tout au moins de leurs démonstrations. C.Q.F.D.

Réciproquement toute classe V normale et compacte est une classe (Δ); *c'est même à votre sens, une classe* (D) *ayant exactement l' "enclosable property."*

En effet, elle satisfait par hypothèse aux conditions 1°, 2°, 3° imposées à Δ. Elle satisfait à 4°. Autrement dit, si tous les éléments limites d'une suite infinie $A_1, A_2, \cdots$ d'éléments de V coïncident, il existe un élément B de V, tel que (B, A_n) tend vers zéro avec $1/n$. En effet, soit B le seul élément limite de la suite. Si (B, A_n) ne tendait pas vers zéro, il existerait un nombre $\epsilon > 0$ et une suite $A_{n_1}, A_{n_2}, \cdots$ extraite de la suite donnée telle que $(B, A_{n_p}) > \epsilon$ quel que soit p. Mais la classe étant compacte (et sans cette supposition la propriété 4° ne serait plus exacte) on peut extraire de $A_{n_1}, A_{n_2}, \cdots$ une suite $A'_{n_1}, A'_{n_2}, \cdots$ qui convergerait vers un élément qui serait précisément B d'après l'hypothèse. Donc les quantités (B, A'_{n_p}) ne resteraient pas toutes supérieures à ϵ.

Enfin notre classe (V) satisfait à la condition (5°) de Δ et même à l' "enclosable property." Il suffit pour cela d'appeler $Q_1(A)$ la classe elle-même et $Q_i(A)$ l'ensemble de l'ensemble $E_{i+1}(A)$ des éléments B de la classe tels que $(A, B) \leq \alpha_{i+1}$ et de leurs éléments limites. Les nombres α_{i+1} sont des

*Rendiconti del Circolo matematico di Palermo, vol. 30 (1910), p. 3, 4.

nombres >0 choisis à l'avance indépendamment de A de la façon suivante. Soit $f(\epsilon)$ la fonction qui entre dans la définition du voisinage et soit $\phi(\epsilon)$ le plus grand des deux nombres dont l'un est ϵ, l'autre étant la borne supérieure de f entre 0 et ϵ. Prenons par exemple $\alpha_2 = 1$; puis en général ayant choisi α_i nous prendrons pour α_{i+1} un nombre >0, inférieur à $\frac{1}{2}\alpha_i$ et tel que $\phi(\phi(\alpha_{i+1})) < \alpha_i$, ce qui est toujours possible puisque $\phi(\epsilon)$ tend vers zéro avec ϵ. Les nombres $\alpha_2, \alpha_3, \cdots$ vont donc en décroissant et tendent vers zéro. De plus les éléments limites C de $E_{i+1}(A)$ vérifient évidemment $(A, C) \leq f(\alpha_{i+1})$. Donc tous les éléments de $Q_{i+1}(A)$ ont avec A un voisinage au plus égal à $\phi(\alpha_{i+1})$. Il est bien évident que $Q_{i+1}(A)$ appartient à $Q_i(A)$ et que les ensembles $Q(A)$ ont un seul élément commun qui est A, enfin que A est intérieur à tous les $Q(A)$. Soit maintenant m un entier quelconque; si on prend n assez petit pour que $\phi(\phi(\alpha_n)) < \alpha_m$, la condition b) de 5° sera vérifiée. En effet, si B, C appartiennent à $Q_n(A)$, on a $(A, B) \leq \phi(\alpha_n)$, $(A, C) \leq \phi(\alpha_n)$, donc $(B, C) \leq \phi(\phi(\alpha_n)) \leq \alpha_m$, c'est à dire que tout élément C de $Q_n(A)$ appartient à $Q_m(B)$.

Pour montrer que notre classe (V) est une classe (Δ), il reste à prouver que les ensembles $Q_i(A)$ sont fermés ce qui est évident puisque $Q_{i+1}(A) = E_{i+1}(A) + E'_{i+1}(A)$.

Mais nous voulons même prouver que V a l' "enclosable property":

1° il faut montrer en outre que $Q_{i+1}(A)$ est intérieur à $Q_i(A)$. Or considérons une suite $B_1, B_2, \cdots$ d'éléments tendant vers un élément B de $Q_{i+1}(A)$; on aurait $(A, B) \leq \phi(\alpha_{i+1})$ et par suite pour n assez grand $(A, B_n) \leq \phi(\phi(\alpha_{i+1})) \leq \alpha_i$, c'est à dire que les B_n appartiendraient à $Q_i(A)$ pour n assez grand.

2° il faut montrer que dans la propriété b), $Q_n(A)$ doit non seulement appartenir à $Q_m(B)$ mais même lui être *intérieur*. Or nous savons choisir n de façon que tout Q_n à qui appartient B appartienne à $Q_{m+1}(B)$; alors il sera intérieur à $Q_m(B)$ d'après ce qui précède.

Enfin nous savons que, pour toute classe V, tout ensemble dérivé est fermé.

Ainsi *nous voyons que la famille des classes* (D) *ayant l' "enclosable property" forme une partie seulement des classes* (V) *normales, à savoir, exactement l'ensemble de celles-ci qui sont compactes.* Or cette limitation qui n'a pas une valeur négligeable dans le cas des ensembles linéaires, devient très importante dans le cas d'espaces fonctionnels plus compliqués. Par exemple, imposer à un ensemble de fonctions continues (dont la classe est (V) normale) la condition d'être compact n'astreint pas à considérer des fonctions seulement bornées dans leur ensemble, mais aussi "également continues" au sens d'Arzelà.

Poitiers
le 3 Janvier, 1912.

IMPLICIT FUNCTIONS DEFINED BY EQUATIONS WITH VANISHING JACOBIAN*

BY

GUY ROGER CLEMENTS

Let

$$y_i = f_i(x_1, \cdots, x_n) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

be n functions, each analytic in the point $(x_1, \cdots, x_n) = (a_1, \cdots, a_n)$, and let

$$f_i(a) = b_i \qquad (i = 1, 2, \cdots, n).$$

The nature of the inverse functions

$$x_i = g_i(y_1, \cdots, y_n)$$

in the neighborhood of the point $(y) = (b)$ is familiar† for the case in which the Jacobian

$$J = \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)},$$

does not vanish in the point $(x) = (a)$, and for the case in which J vanishes identically in the neighborhood of this point. This paper deals with the intermediate case in which J vanishes in the point $(x) = (a)$, but does not vanish identically in the n variables x. Concerning this case, Professor OSGOOD has pointed out that, as examples show, the solution of the system of equations (1) for $x_1, \cdots, x_n$ may lead to multiple-valued functions which are not analytic in the point $(y) = (b)$, and has made the suggestion, "Dies dürfte wohl stets der Fall sein." This turns out in fact to be the case whenever an inverse exists throughout the neighborhood of the point $(x) = (a)$.

Some phases of the problem here suggested have been studied by LONGLEY,‡ DEDERICK,§ and URNER‖ (see also the note to Theorem 4). For the most of their work, f_i is a real function of real variables. For the case treated here f_i

* Presented to the Society, April 28, 1911, and October 26, 1912. See Bulletin, American Mathematical Society, vol. 18 (1912), p. 451.

† OSGOOD, *Encyclopädie der Mathematischen Wissenschaften*, vol. II, p. 104, Sec. 44.

‡ Bulletin of the American Mathematical Society, vol. 17 (1910), p. 1.

§ These Transactions, vol. 14 (1913), p. 143.

‖ Ibid., vol. 13 (1912), p. 232.

is an analytic function of the n complex arguments $x_1, x_2, \cdots, x_n$. For two of the theorems the proof is given only in the case $n = 2$.

If $x_1, x_2, \cdots, x_n$ denote n independent complex variables, any region which contains at least the totality of all points $(x_1, x_2, \cdots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \quad |x_2 - a_2| < \epsilon, \quad \cdots, \quad |x_n - a_n| < \epsilon,$$

for some $\epsilon > 0$, is called a *neighborhood* of the point $(a_1, a_2, \cdots, a_n)$. For purposes of simplicity, the neighborhood of the point $(0, 0, \cdots, 0)$ is considered throughout this paper.

PART I.

THE TRANSFORMAT ON T.

Consider the transformation

$$T: \qquad x = f(u, v), \qquad y = \varphi(u, v),$$

(*a*) *$f(u, v)$ and $\varphi(u, v)$ denoting functions of the complex variables u and v, single-valued and analytic throughout a neighborhood R of $u = 0$, $v = 0$; and satisfying the conditions*

$$(b) \qquad f(0, 0) = 0, \qquad \varphi(0, 0) = 0;$$

$$(c) \qquad J(u, v) = f_u \varphi_v - f_v \varphi_u \not\equiv 0, \qquad J(0, 0) = 0.$$

To any point (u, v) of R, there corresponds, under T, one and but one point (x, y) of the (x, y) space. The totality of points (x, y) which thus correspond to points (u, v) of R, constitute a finite region $\bar{R}$ of the (x, y) space. In general, more than one point (u, v) will yield the same pair of values for x and y. We shall count such a point (x, y) but once, thus regarding it as completely characterized by its coördinates, and seek an inverse transformation

$$u = \bar{f}(x, y), \qquad v = \bar{\varphi}(x, y)$$

which will put in evidence all those points (u, v) of R which correspond under T to any point with coördinates (x, y) of $\bar{R}$. Thus the functions $\bar{f}(x, y)$, $\bar{\varphi}(x, y)$ are to be defined only in the points of $\bar{R}$, and there uniquely or as multiple valued functions.

We shall make repeated use of the following form of the Weierstrassian* preparation theorem:

If $F(y; x_1, \cdots, x_n)$ is analytic in the point $(0; 0, \cdots, 0)$, and if

* WEIERSTRASS, *Abhandlungen aus der Funktionenlehre*, p. 105; PICARD, *Traité d' Analyse*, vol. 2, p. 241; GOURSAT, *Cours d' Analyse*, vol. 2, Sec. 356, p. 280; BLISS, Bulletin of the American Mathematical Society, vol. 16 (1910), p. 356; MACMILLAN, ibid., vol. 17 (1910), p. 116.

$F(0; 0, \cdots, 0) = 0$; if furthermore $F(y; 0, \cdots, 0) \neq 0$, and if $y = 0$ is a zero of order m of $F(y; 0, \cdots, 0)$, then $F(y; x_1, \cdots, x_n)$ can be expressed in the form

$$F(y; x_1, \cdots, x_n) \equiv \Gamma(y; x_1, \cdots, x_n)\, \omega(y; x_1, \cdots, x_n),$$

where $\omega(y; x_1, \cdots, x_n)$ is analytic in the point $(0; 0, \cdots, 0)$ and $\omega(0; 0, \cdots, 0) \neq 0$, and furthermore

$$\Gamma(y; x_1, \cdots, x_n) \equiv y^m + A_1(x_1, \cdots, x_n)y^{m-1} + \cdots + A_m(x_1, \cdots, x_n),$$

$A_i(x_1, \cdots, x_n)$ being analytic in the point $(0, \cdots, 0)$ with $A_i(0, \cdots, 0) = 0$ $(i = 1, 2, \cdots, m)$.

We can assume without essential restriction that $f(u, v)$, $\varphi(u, v)$ and $J(u, v)$ satisfy the conditions of this theorem. That the first two conditions are satisfied follows immediately from the hypotheses concerning T. But suppose, for example, that $f(u, 0) \equiv 0$. Since $f(u, v)$ is analytic in the point $(0, 0)$, it can be written as a series of the form

$$f(u, v) \equiv f_1(u, v) + f_2(u, v) + \cdots + f_i(u, v) + \cdots, \tag{2}$$

where $f_i(u, v)$ is a homogeneous polynomial of degree i in u and v, or vanishes identically. Since by hypothesis, $J(u, v) \not\equiv 0$, it follows that $f(u, v) \not\equiv 0$ and $\varphi(u, v) \not\equiv 0$. Hence there must exist at least one $f_i(u, v) \not\equiv 0$. Suppose j to be the degree of the first $f_i(u, v)$ which is not identically zero. Make the change of variables

$$\begin{aligned} u &= au' + bv', \\ v &= cu' + dv', \end{aligned} \tag{3}$$

subject to the conditions* that

$$ad - bc \neq 0,$$

and that the coefficient of u'^j in the polynomial into which f_j is transformed is not zero.

This coefficient, which has the value $f_j(a, c)$, is a homogeneous polynomial of degree j in a and c with at least one coefficient different from zero. Taking for a and c any two numbers for which this polynomial does not vanish, one can then always determine b and d so that further $ad - bc \neq 0$. In the same way we see that, to insure $\varphi(u, 0) \not\equiv 0$ and $J(u, 0) \not\equiv 0$, it is sufficient, since J is invariant under linear transformation,† to avoid a finite number of values of the ratio $a : c$.

The transformed functions are in each case single-valued and analytic in the neighborhood of $u' = 0$, $v' = 0$, and vanish in this point. Since the

* WEIERSTRASS, loc. cit., p. 113.

† GOURSAT, *Cours d' Analyse*, vol. 1, Sec. 29.

Jacobian determinant for the transformed functions differs from that for the functions $f(u, v)$ and $\varphi(u, v)$ only by the constant factor $ad - bc \neq 0$, the vanishing of the Jacobian in the point $(0, 0)$ has not been affected.

Any function

$$f(u, v) \equiv f_1(u, v) + f_2(u, v) + \cdots + f_i(u, v) + \cdots,$$

which has a coefficient different from zero for the term in u alone in that first polynomial $f_i(u, v)$ which is not identically zero, will be spoken of in the subsequent discussion as *normalized.*

THEOREM 1: *The transformation T can never have an inverse*

$$u = \bar{f}(x, y), \qquad v = \bar{\varphi}(x, y)$$

such that both $\bar{f}(x, y)$ and $\bar{\varphi}(x, y)$ are single-valued and analytic throughout the neighborhood $\bar{R}$ of $x = 0, y = 0$, and vanish when $x = 0, y = 0$.

Suppose such an inverse to exist. Then $\bar{J}(x, y) \equiv \bar{f}_x \bar{\varphi}_y - \bar{f}_y \bar{\varphi}_x$, being a polynomial in the first partial derivatives of functions analytic in $\bar{R}$, is itself analytic in $\bar{R}$. But*

$$J(u, v) \cdot \bar{J}[f(u, v), \varphi(u, v)] \equiv 1, \quad \text{and} \quad J(0, 0) = 0.$$

From this contradiction follows the truth of the theorem.

Both theorem and proof evidently hold for a similar transformation in any number of variables.

THEOREM 2: *If, in the transformation T, $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$, and if the region R be suitably restricted, then there exists an inverse, defined throughout the neighborhood of $x = 0, y = 0$, everywhere continuous in that neighborhood, finitely multiple valued, but not single valued, analytic except along a complex one-dimensional locus, and having the value $u = 0, v = 0$ when $x = 0, y = 0$.*

Suppose the functions $f(u, v)$ and $\varphi(u, v)$ have been normalized and that

$$f_u(0, 0) = \cdots = f_{u^{k-1}}(0, 0) = 0, \qquad f_{u^k}(0, 0) \neq 0;$$

$$\varphi_u(0, 0) = \cdots = \varphi_{u^{i-1}}(0, 0) = 0, \qquad \varphi_{u^i}(0, 0) \neq 0.$$

Consider the equations

$$f(u, v) - x = 0, \tag{4}$$

$$\varphi(u, v) - y = 0$$

with left members analytic in a region S made up of R for u and v, and of the x and y planes.

* JORDAN, *Cours d'Analyse*, vol. 1, 2d ed. (1893), p. 89.

From the Weierstrassian preparation theorem, we have

$$
\begin{aligned}
f(u,v)-x &\equiv [\,u^k + A_1(v,x)\,u^{k-1} + \cdots + A_k(v,x)\,]\,a(u,v,x)\\
&\equiv g(u,v,x)\,a(u,v,x),\\
\varphi(u,v)-y &\equiv [\,u^l + B_1(v,y)u^{l-1} + \cdots + B_l(v,y)\,]\,b(u,v,y)\\
&\equiv h(u,v,y)\,b(u,v,y),
\end{aligned}
\tag{5}
$$

where throughout a suitably chosen neighborhood S' of the point $(0,0,0,0)$, each of the functions g, h, a, b is analytic, and $a(u, v, x) \neq 0$, $b(u, v, y) \neq 0$. For definiteness, suppose h_1 so chosen that

$$|u| < h_1, \quad |v| < h_1, \quad |x| < h_1, \quad |y| < h_1, \quad 0 < h_1,$$

defines such a neighborhood S'. Then, corresponding to any point (v', x', y') in a region

$$\Sigma: \qquad |v| < h_2, \quad |x| < h_2, \quad |y| < h_2, \quad 0 < h_2 \leqq h_1,$$

the equations $f(u, v') - x' = 0$, $\varphi(u, v') - y' = 0$ have respectively exactly k, l roots u for which $|u| < h_1$. These are the roots of the polynomials $g(u, v', x')$ and $h(u, v', x')$ respectively. Since we seek only those roots u of equations (4) for which $|u| < h_1$, when x and y are suitably restricted in absolute value, it therefore will be sufficient to seek the solution of (4) from the equations

$$
\begin{aligned}
g(u,v,x) &\equiv u^k + A_1(v,x)u^{k-1} + \cdots + A_k(v,x) = 0,\\
h(u,v,y) &\equiv u^l + B_1(v,y)u^{l-1} + \cdots + B_l(v,y) = 0.
\end{aligned}
\tag{6}
$$

Compute for this pair of equations the resultant $R(v, x, y)$. In particular we can make this computation by means of Euclid's algorithm* or Sylvester's determinant. The identical vanishing of $R(v, x, y)$ in v, x and y is the necessary and sufficient condition that $g(u, v, x)$ and $h(u, v, y)$ have a common factor throughout Σ. But $R(v, 0, 0) \not\equiv 0$ and hence, a fortiori, $R(v, x, y) \not\equiv 0$. For if $R(v, 0, 0) \equiv 0$, then $g(u, v, 0)$ and $h(u, v, 0)$ must have a common factor in the point $(0, 0, 0, 0)$. Since $a(u, v, 0) \neq 0$, any factor of $f(u, v)$ must be a factor of $g(u, v, 0)$ and vice versa from identities 5. Similarly, any factor of $\varphi(u, v)$ must be a factor of $h(u, v, 0)$ and vice versa. By hypothesis, f and φ have no common factor in the point $(0, 0)$. Therefore $g(u, v, 0)$ and $h(u, v, 0)$ have no common factor in the point $(0, 0, 0, 0)$ and consequently $R(v, 0, 0) \not\equiv 0$.

Since $R(v, x, y)$ is a polynomial in the variables A_i, B_j with vanishing constant term, and since the analytic functions A_i, B_j $(i = 1, 2, \cdots, k;$

* See BÔCHER'S *Algebra*, Sec. 70, Sec. 76.

$j = 1, 2, \cdots, l$) vanish in the point $(0, 0)$, it follows that $R(v, x, y)$ is an analytic function of v, x and y in Σ, which vanishes in the point $(0, 0, 0)$; that is,

$$R(v, x, y) \text{ is analytic in } \Sigma;$$

$$R(0, 0, 0) = 0;$$

$$R(v, 0, 0) \not\equiv 0.$$

Therefore, $R(v, x, y)$ satisfies all the conditions of the Weierstrassian preparation theorem; and if $v = 0$ is a zero of order n of $R(v, 0, 0)$, then, in a neighborhood

$$\Sigma': \qquad |v| < h_3, \quad |x| < h_3, \quad |y| < h_3, \quad 0 < h_3 \leqq h_2,$$

$$(7)\quad R(v, x, y) \equiv [v^n + C_1(x, y)v^{n-1} + \cdots + C_n(x, y)]\, c(v, x, y)$$
$$\equiv G(v, x, y)\, c(v, x, y),$$

where C_i is analytic in $(0, 0)$, $C_i(0, 0) = 0$ $(i = 1, 2, \cdots, n)$ and $c(v, x, y)$ is analytic in $(0, 0, 0)$ and does not vanish there. Further, for any point (x', y') of a certain region

$$\sigma: \qquad |x| < h_4, \quad |y| < h_4, \quad 0 < h_4 \leqq h_3$$

the equation $R(v, x', y') = 0$ has exactly n roots v for which $|v| < h_3$, and these are the n roots of the polynomial

$$v^n + C_1(x', y')v^{n-1} + \cdots + C_n(x', y').$$

But v, thus defined, is an n-valued continuous function of the coefficients $C_1, \cdots, C_n$,* and consequently an n-valued continuous function of x and y in the point (x', y').

What is the behavior of this solution as (x, y), starting from (x', y') is allowed to vary continuously in the region σ?
If $G(v, x, y)$ is reducible, i. e., if

$$G(v, x, y) \equiv [G_1(v, x, y)]^{l_1} \cdots [G_k(v, x, y)]^{l_k},$$

where $G_i(v, x, y)$ is analytic in $(0, 0, 0)$ and $G_i(0, 0, 0) = 0$ $(i = 1, 2, \cdots, k)$ then the solution of the equation $G(v, x, y) = 0$ evidently can be built up from the solution of the equations

$$G_1(v, x, y) = 0, \ \cdots, \ G_k(v, x, y) = 0.$$

* Cauchy, *Turin Memoirs*, 1831; Coolidge, Annals of Mathematics, second series, vol. 9 (1908), p. 116; Weber, *Lehrbruch der Algebra*, Sec. 44; Briot et Bouquet, *Theorie des fonctions elliptiques*, Chapter 3, p. 31.

Any $G_i(v, x, y)$ can be written in the form

$$v^m + a_1(x, y)\, v^{m-1} + \cdots + a_m(x, y),$$

where $a_i(x, y)$ has properties identical with those enumerated above for $C_i(x, y)$. Consequently we limit our immediate attention to an equation

$$M(v, x, y) \equiv v^m + a_1(x, y)\, v^{m-1} + \cdots + a_m(x, y) = 0, \tag{8}$$

where $M(v, x, y)$ is irreducible, $a_1(x, y), \cdots, a_m(x, y)$ are analytic in σ, and $a_1(0, 0) = 0, \cdots, a_m(0, 0) = 0$.

If $m = 1$, the equation (8) defines v throughout σ as a single-valued and analytic function of x and y. If $m > 1$, form the discriminant $D(x, y)$ of $M(v, x, y)$. $D(x, y)$ is a polynomial in $a_1(x, y), \cdots, a_m(x, y)$ with no constant term, and consequently a single-valued and analytic function of x and y for which $D(0, 0) = 0$. Since $M(v, x, y)$ is irreducible, $D(x, y) \not\equiv 0$. Each of the equations obtained by setting equal to zero an irreducible factor of $D(x, y)$ defines a continuous locus in the (x, y) space. Corresponding to every point (x_0, y_0) for which $D(x_0, y_0) \neq 0$, the equation $M(v, x_0, y_0) = 0$ has m distinct roots, $v_1, v_2, \cdots, v_m$. Consequently $M(v, x, y) = 0$ determines m functions $v(x, y)$, each a single-valued and analytic function of x and y throughout a suitably restricted neighborhood of $x = x_0$, $y = y_0$, this region in no case extending to a point in which $D(x, y) = 0$.

Let $(\bar{x}, \bar{y})$ be any point (x, y) in σ, for which $D(x, y) = 0$. The m roots of the polynomial

$$M(v, \bar{x}, \bar{y}) \equiv v^m + a_1(\bar{x}, \bar{y})\, v^{m-1} + \cdots + a_m(\bar{x}, \bar{y})$$

are each less than h_3 in absolute value. Denote them by $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_m$.

Suppose $\bar{v}_1$ is a simple root; then as above, $M(v, x, y) = 0$ has a solution v, which is an analytic function of x and y in the neighborhood of $x = \bar{x}$, $y = \bar{y}$, and takes the value $\bar{v}_1$ in this point. This solution can be extended analytically* at least to any point of σ which is not a point of the locus $D(x, y) = 0$.

Since $D(\bar{x}, \bar{y}) = 0$, at least two of the $\bar{v}$'s are equal. Suppose $\bar{v}_2$ is a root of order k, $1 < k \leqq m$. Give to $\bar{x}$ and $\bar{y}$ increments $\Delta\bar{x}$ and $\Delta\bar{y}$, subject to the single restriction

$$|\bar{x} + \Delta\bar{x}| < h_4, \qquad |\bar{y} + \Delta\bar{y}| < h_4.$$

Then $a_1(\bar{x}, \bar{y}), \cdots, a_m(\bar{x}, \bar{y})$ receive increments $\Delta a_1, \cdots, \Delta a_m$. Since $a_i(x, y)$ is an analytic function of x and y, we can so control $\Delta\bar{x}$ and $\Delta\bar{y}$ that Δa_i shall be arbitrarily small in absolute value. Consequently, if ϵ be an arbitrarily assigned small positive quantity, it is always possible to

* *Encyklopädie der Mathematischen Wissenschaften*, vol. II, p. 108, Sec. 47.

determine a value δ, depending on ϵ, so that if

$$|\Delta a_i| < \delta \qquad (i = 1, 2, \cdots, m)$$

the equation

$$v^m + [a_1(\bar{x}, \bar{y}) + \Delta a_1] v^{m-1} + \cdots + a_m(\bar{x}, \bar{y}) + \Delta a_m = 0$$

has k and only k roots,* $\bar{v}'$, $\bar{v}''$, $\cdots$, $\bar{v}^{(k)}$, for which

$$(9) \qquad |\bar{v}_2 - \bar{v}'| < \epsilon, \quad |\bar{v}_2 - \bar{v}''| < \epsilon, \quad \cdots, \quad |\bar{v}_2 - \bar{v}^{(k)}| < \epsilon.$$

That is, k of the solutions v can be grouped so as to form a k-valued continuous function of x and y in the neighborhood of the point $(\bar{x}, \bar{y})$. If, further, $\Delta\bar{x}$ and $\Delta\bar{y}$ are chosen so that $D(\bar{x} + \Delta\bar{x}, \bar{y} + \Delta\bar{y}) \neq 0$, then, as we have seen, in the neighborhood of any such point $(\bar{x} + \Delta\bar{x}, \bar{y} + \Delta\bar{y})$, the solutions v can be grouped so as to yield m single-valued and analytic functions of x and y.

Now let $\Delta\bar{x}$ and $\Delta\bar{y}$ approach zero while $D(\bar{x} + \Delta\bar{x}, \bar{y} + \Delta\bar{y}) \neq 0$. The inequalities (9) show that there are k and only k of these m roots v which, when (x, y) approaches $(\bar{x}, \bar{y})$, approach as their limiting value $\bar{v}_2$, the k-fold value of v when $x = \bar{x}$, $y = \bar{y}$. The other $m - k$ roots can be grouped similarly with regard to $\bar{v}_1$, $\bar{v}_3$, $\cdots$, $\bar{v}_m$. But $(\bar{x}, \bar{y})$ is any point of the locus $D(x, y) = 0$. Therefore the roots have this behavior in every point of $D(x, y) = 0$. That is, in the neighborhood of any point (x, y) not on $D(x, y) = 0$ there are defined m roots v, each of which is a single-valued and analytic function of x and y in this point; as (x, y) approaches a point of the locus $D(x, y) = 0$, each root remains continuous and has as its limiting value some one of the roots at that point. Therefore $M(v, x, y) = 0$ defines v throughout σ as an m-valued, continuous function of x and y, analytic, with m distinct branches, except in points of the complex one-dimensional locus $D(x, y) = 0$, where it is continuous and less than m valued. Since $M(v, 0, 0) = 0$ has an m-tuple root $v = 0$, every element of this m-fold continuous function has the value zero when $x = 0$, $y = 0$.

This discussion is typical for any of the equations $G_1(v, x, y) = 0$, $\cdots$, $G_k(v, x, y) = 0$. Let h denote the smallest of the k numbers h_i, and H the largest of the h_i. Then, throughout the region

σ': $$|x| < h, \qquad |y| < h,$$

the equation

$$G(v, x, y) \equiv [G_1(v, x, y)]^{l_1} \cdots [G_k(v, x, y)]^{l_k} = 0$$

defines v as a function of x and y for which $|v| < H$, and which moreover is

* See the reference above to CAUCHY and others. See NEUMANN, *Abelsche Integrale*, 2d ed., (1884), p. 125, for a definition of continuity for multiple-valued functions.

everywhere continuous, not more than n-valued, analytic except along a complex one-dimensional locus, and has the value zero when $x = 0$, $y = 0$. If G_i is a polynomial in v of degree m_i, then $n = \sum_{i=1}^{k} l_i m_i$.

We saw above that in the neighborhood of the origin, this equation $G(v, x, y) = 0$ solves completely the equation $R(v, x, y) = 0$. That is, if we choose arbitrarily a point (x', y') in σ', there will always be at least one point (v, x', y') in Σ' for which $R(v, x', y') = 0$. But in every point (v, x, y) of Σ' for which $R(v, x, y) = 0$, the left members of (4), which when the coördinates of this point are substituted become functions of u alone, have a common factor, thus defining for u a value or values in the point (x', y').

Since u and v enter symmetrically in the foregoing discussion, the results apply when u and v are interchanged. Thus, without making use of any assumption concerning the value of $J(u, v)$ in R, we have shown the existence of functions $u(x, y)$ and $v(x, y)$ which in the region σ' fulfill the demands of the theorem, with a single possible exception.

May this inverse not be single-valued for some transformation T satisfying the hypotheses of Theorem 2? If the inverse is single-valued, v must be a single-valued function of x and y; hence in the equation (8) which defines v, m must have the value one. But if $m = 1$, v is a single-valued and *analytic* function of x and y throughout the neighborhood of $x = 0, y = 0$. Similarly, if the inverse is single-valued u must be a single-valued and *analytic* function of x and y throughout the neighborhood of $x = 0, y = 0$. Both $u(x, y)$ and $v(x, y)$ have the value zero when $x = 0, y = 0$. This is in direct contradiction to Theorem 1. Hence Theorem 2 is proved.

If instead of the transformation T we consider the transformation T', whose definition differs from that for T in that it omits the condition (c), we deduce the following theorem:

THEOREM 3: *If to each point (x, y) of the region R there corresponds in consequence of T' a finite number of points (u, v) in a suitably restricted region R, and if to some point (x, y) of R there corresponds more than one point (u, v) of R, then there exists an inverse, defined throughout the neighborhood of $x = 0$, $y = 0$, everywhere continuous in that neighborhood, finitely multiple-valued, but not single-valued, analytic except along a complex one-dimensional locus, and having the value $u = 0, v = 0$ when $x = 0, y = 0$.*

If to each point of the region R there corresponds in consequence of T' a single point (u, v) of the region R, then u and v are analytic in x and y in the point $x = 0, y = 0$.

The functions $f(u, v)$ and $\varphi(u, v)$ can have no common factor in the point $(0, 0)$, for if they did, the locus defined by equating to zero this common factor

would correspond to the point $x = 0, y = 0$, contrary to the hypothesis that to each point (x, y) of $\bar{R}$ there correspond only a finite number of points (u, v) of R. In particular, neither $f(u, v)$ nor $\varphi(u, v)$ can be identically zero.

We show next, that because of this condition that to each point (x, y) there shall correspond a finite number of points (u, v) it follows that $J(u, v) \not\equiv 0$. Since $f(u, v)$ is not a constant, there must be a point (u_1, v_1) in R at which a first partial derivative of $f(u, v)$ with respect to one of its arguments, say $f_u(u, v)$, is not zero. If $J(u, v) \equiv 0$, then in a suitable neighborhood of $x = x_1 = f(u_1, v_1)$, y is expressible as an analytic function of x alone.* Consequently the same point (x, y) will correspond to every point (u, v) which is a point of the locus $f(u, v) = x_1$ and is within a suitable neighborhood S of (u_1, v_1) in which $f_u(u, v) \neq 0$. This would set up a one-to-infinity correspondence, contrary to hypothesis. Therefore $J(u, v) \not\equiv 0$.

Two cases must be considered: $J(0, 0) = 0$, and $J(0, 0) \neq 0$. If $J(0, 0) = 0$, all the conditions of Theorem 2 are fulfilled. Consequently the inverse must be multiple-valued and must satisfy all the conclusions of the first paragraph of Theorem 3. If $J(0, 0) \neq 0$, the transformation T' defines u and v as analytic in x and y in the point $(0, 0)$ and sets up a one-to-one relation between suitably chosen neighborhoods of the point $(x, y) = (0, 0)$ and the point $(u, v) = (0, 0)$. Hence the theorem is proved.

It follows from Theorem 1 *and from Theorem* 3 *that there can not exist a transformation* T *for which the inverse is single-valued throughout any neighborhood* $\bar{R}$ *of the point* $x = 0, y = 0$.

Part II.

More General Systems of Equations.

Theorem 4:† *The system of equations*

* Jordan, *Cours d'Analyse*, Vol. I, 2d ed. (1893), p. 86.

† This theorem is a generalization of the Weierstrass preparation theorem. It was developed without knowledge of the important results published in these Transactions for April, 1912, by G. A. Bliss under the title *A generalization of Weierstrass's preparation theorem for power series in several variables*, and it differs from the principal theorem of that paper in its hypotheses and consequently in the scope of its application.

Professor Bliss considers certain "characteristic" polynomials, $g_m(y_1, \cdots, y_p)$, namely the homogeneous polynomials of lowest degree in the series $f_m(0, \cdots, 0; y_1, \cdots, y_p)$, and discusses the solution of such a set of equations as we are considering under the hypothesis that the resultant R of these polynomials g_m be different from zero. This is a condition sufficient to establish results with which, except in a simple case, the results of my Theorem 4 are identical. My hypotheses impose no condition on the value of R, but my condition $J_k(0, \cdots, 0; 0, \cdots, 0) \neq 0$, demands that $p - 1$ of the characteristic polynomials g_m be linear and that at least one determinant of their matrix be different from zero. It was erroneously assumed by Dr. L. L. Dines in a paper presented to the American Mathematical Society on October 26, 1912 (see Bulletin, vol. 19, p. 165) that I had required in all cases $R \neq 0$. In the case $R \neq 0$, my theorem is a special case of that announced by Professor Bliss.

$$
\begin{aligned}
&f_1(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,\\
&f_2(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,\\
&\quad\cdots\cdots\cdots\cdots\\
&f_p(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,
\end{aligned}
\tag{10}
$$

where

(*a*) f_i $(i = 1, 2, \cdots, p)$ *is an analytic function of its arguments throughout a neighborhood* R *of* $(x) = 0$, $(y) = 0$;

(*b*) $$f_i(0, \cdots, 0; 0, \cdots, 0) = 0;$$

(*c*) $$J_1 \equiv \frac{\partial(f_1, f_2, \cdots, f_p)}{\partial(y_1, y_2, \cdots, y_p)} = 0 \qquad \textit{when } (x) = 0,\ (y) = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$J_{k-1} \equiv \frac{\partial(J_{k-2}, f_2, \cdots, f_p)}{\partial(y_1, y_2, \cdots, y_p)} = 0 \qquad \textit{when } (x) = 0,\ (y) = 0,$$

$$J_k \equiv \frac{\partial(J_{k-1}, f_2, \cdots, f_p)}{\partial(y_1, y_2, \cdots, y_p)} \neq 0 \qquad \textit{when } (x) = 0,\ (y) = 0,$$

defines $y_1, y_2, \cdots, y_p$ *as functions of* $x_1, x_2, \cdots, x_n$, *continuous throughout the neighborhood of* $(x) = 0$. *With a suitable counting of multiplicities always present this solution is* k*-fold; it is analytic with the possible exception of complex* $(n-1)$*-dimensional loci where the roots, in general distinct, become coincident. When* (x) *is zero,* (y) *is zero.*

If $p = 1$, this theorem is the Weierstrassian preparation theorem; for the conditions then become $f_1(x_1, \cdots, x_n; y_1) = 0$, where $f_1(0, \cdots, 0; 0) = 0$; $f_{1y_1}(0, \cdots, 0; 0) = \cdots = f_{1y_1^{k-1}}(0, \cdots, 0; 0) = 0$; $f_{1y_1^k}(0, \cdots, 0; 0) \neq 0$. Hence, throughout a complete neighborhood of the point $(0, \cdots, 0)$, the equation $f_1(x_1, \cdots, x_n; y_1) = 0$ defines y_1 as a function of $x_1, \cdots, x_n$, either analytic throughout the region, or everywhere continuous and analytic except along certain loci where the roots, in general distinct, become coincident. If we count multiple roots in the usual way, this solution is k-valued.

On the other hand, when the restriction is made that $f_2, \cdots, f_p$ have linear characteristic polynomials, the results of Professor Bliss' theorem follow from mine. Thus the theorems overlap while neither is a special case of the other.

At the end of Theorem 4, I give two examples neither of which satisfies the condition $R \neq 0$, for "if the characteristic polynomials are all of degree one, their determinant is the resultant R" (Bliss, loc. cit., p. 136), and in each case this determinant is readily seen to be zero. An example satisfying my Theorem 5, but one for which $R = 0$, is

$$y_2 + y_1 y_3 + y_1^3 - x_1 = 0,$$
$$y_2 - x_2 = 0, \qquad y_3 - x_3 = 0;$$

for which $J_1 \equiv y_3 + 3y_1^2$, $J_2 \equiv 6y_1$, $J_3 \equiv 6$.

The condition $J_k(0, \cdots, 0; 0, \cdots, 0) \neq 0$ requires that at least one determinant from the matrix

$$\left\| \begin{matrix} f_{2y_1}, & \cdots, & f_{2y_p} \\ \cdot & \cdot & \cdot \\ f_{py_1}, & \cdots, & f_{py_p} \end{matrix} \right\| \tag{11}$$

shall be different from zero when $(x) = 0, (y) = 0$. For definiteness suppose the notation so chosen that

$$\frac{\partial(f_2, \cdots, f_p)}{\partial(y_2, \cdots, y_p)} \neq 0, \tag{12}$$

when $(x) = 0$, $(y) = 0$. Then the last row of this determinant,

$$f_{py_2}, \cdots, f_{py_p},$$

must have at least one element which is not zero at the origin. Suppose the notation so chosen that

$$f_{py_p}(0, \cdots, 0; 0, \cdots, 0) \neq 0. \tag{13}$$

Then we can write

$$\begin{aligned} &f_p(x_1, \cdots, x_n; y_1, \cdots, y_p) \\ &\equiv [y_p - A(x_1, \cdots, x_n; y_1, \cdots, y_{p-1})]K(x_1, \cdots, x_n; y_1, \cdots, y_p), \end{aligned} \tag{14}$$

where $K \neq 0$ in a suitable neighborhood of the origin.

Replacing y_p by $A(x_1, \cdots, x_n; y_1, \cdots, y_{p-1})$ in all the equations of the set (10), we are led to the case of $p - 1$ equations

$$\begin{aligned} f_1(x_1, \cdots, x_n; y_1, \cdots, y_{p-1}, A) &= F_1(x_1, \cdots, x_n; y_1, \cdots, y_{p-1}) = 0, \\ \cdot \quad \cdot \quad \cdot \quad

$$\begin{vmatrix} f_{1y_1} & \cdots & f_{1y_i} & \cdots & f_{1y_p} \\ f_{2y_1} & \cdots & f_{2y_i} & \cdots & f_{2y_p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (y_p-A)K_{y_1}-KA_{y_1} & \cdots & (y_p-A)K_{y_i}-KA_{y_i} & \cdots & (y_p-A)K_{y_p}+K \end{vmatrix}$$

which is equal to

$$(16)\quad (y_p-A)\begin{vmatrix} f_{1y_1}, & \cdots & f_{1y_p} \\ \cdots & \cdots & \cdots \\ f_{p-1,\,y_1} & \cdots & f_{p-1,\,y_p} \\ K_{y_1} & \cdots & K_{y_p} \end{vmatrix} + \begin{vmatrix} f_{p-1,\,y_1} & \cdots & f_{1y_{p-1}} & f_{1y_p} \\ \cdots & \cdots & \cdots & \cdots \\ f_{p-1,\,y_1} & \cdots & f_{p-1,\,y_{p-1}} & f_{p-1,\,y_p} \\ -A_{y_1} & \cdots & -A_{y_{p-1}} & 1 \end{vmatrix} K.$$

Set

$$(17)\quad H_1 \equiv \begin{vmatrix} f_{1y_1}, & \cdots, & f_{1y_{p-1}}, & f_{1y_p} \\ \cdots & \cdots & \cdots & \cdots \\ f_{p-1,\,y_1} & \cdots & \cdots & f_{p-1,\,y_p} \\ -A_{y_1}, & \cdots, & -A_{y_{p-1}}, & 1 \end{vmatrix} \equiv \frac{\partial(f_1, f_2, \cdots, f_p, y_p - A)}{\partial(y_1, y_2, \cdots, y_p)},$$

and

$$(18)\quad H_{i+1} \equiv \frac{\partial(H_i, f_2, \cdots, f_{p-1}, y_p - A)}{\partial(y_1, y_2, \cdots, y_{p-1}, y_p)}.$$

In the determinant H_1 multiply the elements of the last column by the lowest element of the ith column, and subtract from the ith column ($i = 1, 2, \cdots, p-1$). Then

$$H_1 \equiv \begin{vmatrix} f_{1y_1} & +f_{1y_p}A_{y_1}, & \cdots, & f_{1y_{p-1}} & +f_{1y_p}A_{y_{p-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{p-1,\,y_1}+f_{p-1,\,y_p}A_{y_1}, & & \cdots, & f_{p-1,\,y_{p-1}}+f_{p-1,\,y_p}A_{y_{p-1}} \end{vmatrix}.$$

That is, H_1 is the function which under the substitution $y_p = A$ reduces to the Jacobian $\bar{J}$ for the $(p-1)$ equations (15). Consequently H_{i+1} goes over into $\bar{J}_{i+1}$ under the same substitution. We have seen that when we replace f_p by $(y_p - A)K$,

$$(18a)\quad J = (y_p - A)\,|f_{1y}\cdots f_{p-1,\,y_{p-1}}K_{y_p}| + KH_1.$$

Suppose it has been shown that

$$(19)\quad J_n = (y_p - A)L + MH_1 + NH_2 \cdots + QH_{n-1} + K^n H_n.$$

We will now show that for any n

$$J_{n+1} \equiv (y_p - A)L' + M'H_1 + \cdots + Q'H_n + K^{n+1}H_{n+1},$$

where $L, M, \cdots, L', M', \cdots$ denote analytic functions of (x) and (y) whose particular form is of no interest. By definition,

$$J_{n+1} \equiv \begin{vmatrix} J_{ny_1} & \cdots & J_{ny_p} \\ f_{2y} & \cdots & \\ \cdots & \cdots & f_{p-1,\,y_p} \\ f_{py_1} & \cdots & f_{py_p} \end{vmatrix}.$$

To replace f_p by $(y_p - A)K$ affects only the first and last rows. If we leave the first row unchanged for the moment, and replace f_p by $(y_p - A)K$ in the last row, we have

$$J_{n+1} \equiv (y_p - A) \begin{vmatrix} J_{ny_1} & \cdots & J_{ny_p} \\ \cdots & \cdots & \cdots \\ f_{p-1,\,y_1} & \cdots & \\ K_{y_1} & \cdots & K_{y_p} \end{vmatrix} + K \begin{vmatrix} J_{ny_1} & \cdots & & J_{ny_p} \\ \cdots & \cdots & \cdots & \cdots \\ f_{p-1,\,y_1} & \cdots & & \\ -A_{y_1} & \cdots & -A_{y_{p-1}} & 1 \end{vmatrix}.$$

We lay aside the first term, which admits $y_p - A$ as a factor, and in the second determinant replace J_n by its value from (19). The elements of the first row will be

$$(y_p - A)L_{y_i} - LA_{y_i} + H_1 M_{y_i} + H_{1y_i} M + \cdots + nK^{n-1} K_{y_i} H_n + K^n H_{ny_i}$$

($i = 1, 2, \cdots, p$, where A_{y_p} is to be replaced by -1).

Expand this determinant in terms of the minors of its first row, then regroup into the sum of the $2n + 2$ determinants of order p

$$\begin{aligned} (20)\quad (y_p - A) &\begin{vmatrix} L_{y_1} & \cdots & L_{y_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix} + L \begin{vmatrix} -A_{y_1} - A_{y_2} & \cdots & 1 \\ \cdots & \cdots & \cdots \\ -A_{y_1} - A_{y_2} & \cdots & 1 \end{vmatrix} \\ + H_1 &\begin{vmatrix} M_{y_1} & \cdots & M_{y_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix} + M \begin{vmatrix} H_{1y_1} & \cdots & H_{1y_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix} \\ + H_2 &\begin{vmatrix} N_{y_1} & \cdots & N_{y_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix} + N \begin{vmatrix} H_{2y_1} & \cdots & H_{2y_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix} \\ \cdots\quad &\cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots \\ + nK^{n-1}H_n &\begin{vmatrix} K_{y_1} & \cdots & K_{y_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix} + K^n \begin{vmatrix} H_{ny_1} & \cdots & H_{ny_p} \\ \cdots & \cdots & \cdots \\ -A_{y_1} & \cdots & 1 \end{vmatrix}. \end{aligned}$$

Of the determinants in the right-hand column, the first is identically zero, the second is H_2, the third is H_3, $\cdots$, the $(n+1)$st is H_{n+1}. Therefore, if the identity (19) is true, then also is

$$J_{n+1} \equiv (y_p - A)L' + M'H_1 + N'H_2 + \cdots + Q'H_n + K^{n+1}H_{n+1}.$$

But (18a) establishes this relation for $n = 0$; it is therefore true for all values of n.

Since $J_1, \cdots, J_{k-1}$, and $y_p - A$ are zero at the origin, and since $K \neq 0$, $J_k \neq 0$ at that point, it follows from these k identities considered for the values $n = 0, \cdots, n = k - 1$ successively, that $H_1 = 0$, $H_2 = 0, \cdots, H_{k-1} = 0$, $H_k \neq 0$, when $(x) = 0$, $(y) = 0$. Consequently the k functions $\bar{J}_1, \bar{J}_2, \cdots, \bar{J}_k$ into which the substitution $y_p = A$ carries the H's satisfy the conditions

$$\bar{J}_1 = 0, \quad \bar{J}_2 = 0, \quad \cdots, \quad \bar{J}_{k-1} = 0, \quad \bar{J}_k \neq 0,$$

when

$$(x) = 0, \quad y_1 = 0, \quad \cdots, \quad y_{p-1} = 0.$$

This process depends in no essential way on the number p of the theorem. We have rather shown that from any system of equations ($p > 1$) satisfying the hypotheses of Theorem 4, it is always possible to build up a second system, one less in number, satisfying exactly analogous hypotheses. Consequently, if we carry through this process $p - 1$ times, we replace the set of equations (10) by the new set

$$\begin{aligned}
y_p &= A_0(y_1, \cdots, y_{p-1}; x_1, \cdots, x_n), \\
y_{p-1} &= A_1(y_1, \cdots, y_{p-2}; x_1, \cdots, x_n), \\
&\cdots\cdots\cdots\cdots\cdots \\
y_3 &= A_{p-3}(y_1, y_2; x_1, \cdots, x_n), \\
y_2 &= A_{p-2}(y_1; x_1, \cdots, x_n), \\
0 &= F(y_1; x_1, \cdots, x_n),
\end{aligned} \tag{21}$$

where A_m ($m = 0, 1, \cdots, p - 2$) and F are single-valued and analytic functions of their arguments in the origin and vanish there, and where further

$$F_{y_1}(0; 0, \cdots, 0) = \cdots = F_{y_1^{k-1}}(0; 0, \cdots, 0) = 0, \quad F_{y_1^k}(0; 0, \cdots, 0) \neq 0.$$

In A_{p-3} replace y_2 by $A_{p-2}(y_1; x_1, \cdots, x_n)$; then in A_{p-4} replace y_2 and y_3 by their values in y_1 and (x), and so on to A_0. The system (21) is thus replaced by the equivalent simpler system

$$\begin{aligned}
y_2 &= A_{p-2}(y_1; x_1, \cdots, x_n) &&= a_2(y_1; x_1, \cdots, x_n), \\
y_3 &= A_{p-3}(y_1, a_2; x_1, \cdots, x_n) &&= a_3(y_1; x_1, \cdots, x_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_{p-1} &= A_1(y_1, a_2, \cdots, a_{p-2}; x_1, \cdots, x_n) &&= a_{p-1}(y_1; x_1, \cdots, x_n), \\
y_p &= A_0(y_1, a_2, \cdots, a_{p-1}; x_1, \cdots, x_n) &&= a_p(y_1; x_1, \cdots, x_n), \\
& F(y_1; x_1, \cdots, x_n) = 0.
\end{aligned} \tag{22}$$

The existence and the uniqueness of the equations $y_m = a_m$ $(m = 2, \cdots, p)$ is a direct consequence* of condition (12), but the method we have followed further establishes a set of working conditions on the function $F(y_1; x_1, \cdots, x_n)$. If we replace F by the equivalent Weierstrassian polynomial of degree k in y_1, then y_1 is defined by the equation

$$r(y_1; x_1, \cdots, x_n) \equiv y_1^k + S_1(x_1, \cdots, x_n)\, y_1^{k-1} + \cdots + S_k(x_1, \cdots, x_n) = 0. \tag{23}$$

In every point of the region of definition which is not a point of the locus defined by equating to zero the discriminant of some irreducible factor of $r(y_1; x_1, \cdots, x_n)$, each of the roots y_1 of the equation $r = 0$ can be expressed as a single-valued and analytic function of $x_1, \cdots, x_n$, and hence by an immediate substitution $y_2, \cdots, y_p$ are determined to be single-valued and analytic in $x_1, \cdots, x_n$ in such a point. If we count possible multiple roots in the usual way, the solution is k-valued. If in the point $(0; 0, \cdots, 0)$, $r(y_1; x_1, \cdots, x_n)$ is completely reducible into factors linear in y_1, the solution is analytic throughout the neighborhood of this point; but in general r is not thus reducible. We have, however, discussed in detail the character of y_1 as defined by means of equation (23). It follows further, from a theorem due to Poincare,† that y_m must also satisfy an equation

$$y_m^k + Q_{m1}(x_1, \cdots, x_n)\, y_m^{k-1} + \cdots + Q_{mk}(x_1, \cdots, x_m) = 0 \quad (m = 2, 3, \cdots, p),$$

where $Q_{mi}(x_1, \cdots, x_n)$ $(m = 2, \cdots, p;\ i = 1, \cdots, k)$ is analytic in the point $(x_1, \cdots, x_n) = (0, \cdots, 0)$ and vanishes there. Therefore throughout a suitable neighborhood of $(x_1, \cdots, x_n) = (0, \cdots, 0)$, y_m is defined as a continuous function of $x_1, \cdots, x_n$, analytic with the possible exception of a complex $(n-1)$-dimensional locus, defined by equating to zero the discriminant of some irreducible factor of $r(y_1; x_1, \cdots, x_n)$, where roots elsewhere distinct become coincident. Furthermore $y_m = 0$ when $(x_1, \cdots, x_n) = (0, \cdots, 0)$ $(m = 1, 2, \cdots, p)$. Hence the theorem is proved.

A simple example of a set of equations satisfying the conditions of this theorem and having solutions everywhere analytic is the system

$$y_1^2 - y_2 - 2y_3 - y_4 = 0, \qquad y_2 - x_1^2 = 0,$$
$$y_3 - x_1 x_2 = 0, \qquad y_4 - x_2^2 = 0;$$

for which

$$J_1 \equiv 2y_1, \qquad J_2 \equiv 2.$$

The solution is

$$y_1 = x_1 + x_2 \quad \text{or} \quad y_1 = -x_1 - x_2$$

* Picard, *Traité d'Analyse*, 2d ed. (1905), vol. 2, p. 267.

† Poincaré, *Thesis*, Paris, 1879, p. 13, Cor. II.

with the functions

$$y_2 = x_1^2, \qquad y_3 = x_1 x_2, \qquad y_4 = x_2^2.$$

A second example is the system

$$y_1^2 + y_2 + 2y_3 + y_4 - 2x_1 y_1 - 2x_2 y_1 = 0,$$

$$y_2 - x_1^2 = 0, \qquad y_3 - x_1 x_2 = 0, \qquad y_4 - x_2^2 = 0,$$

for which $J_1 \equiv 2(y_1 - x_1 - x_2)$, $J_2 \equiv 2$. The inverse is given by the system of functions

$$y_1 = x_1 + x_2, \qquad y_2 = x_1^2, \qquad y_3 = x_1 x_2, \qquad y_4 = x_2^2,$$

counted twice.

Note that in the problems as set, every equation has as its left member an irreducible function of its arguments.

The following theorem gives an example of a further and more general type of problem.

THEOREM 5: *The transformation*

$$\begin{aligned} x_1 &= f_1(y_1, \cdots, y_n), \\ x_2 &= f_2(y_1, \cdots, y_n), \\ &\cdots\cdots\cdots \\ x_n &= f_n(y_1, \cdots, y_n), \end{aligned} \tag{24}$$

where

(*a*) *f_i is a single valued and analytic function of $y_1, \cdots, y_n$ in the point $(0, \cdots, 0)$ and vanishes there $(i = 1, 2, \cdots, n)$;*

(*b*)
$$J_1(0, \cdots, 0) = \frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}\bigg|_{(y)=0} = 0,$$

$$\cdots\cdots\cdots$$

$$J_{k-1}(0, \cdots, 0) = \frac{\partial(J_{k-2}, f_2, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}\bigg|_{(y)=0} = 0,$$

$$J_k(0, \cdots, 0) = \frac{\partial(J_{k-1}, f_2, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}\bigg|_{(y)=0} \neq 0 \qquad (k > 1);$$

has a k valued continuous inverse, defined throughout the neighborhood of $(x) = 0$. This inverse is analytic, with k distinct branches, except along a complex $(n-1)$-dimensional locus, where it is continuous and less than k valued. Finally $(y) = 0$ when $(x) = 0$.

If we consider the set of equations

$$f_1(y_1, \cdots, y_n) - x_1 = 0, \quad \cdots, \quad f_n(y_1, \cdots, y_n) - x_n = 0, \tag{25}$$

in an arbitrary complete neighborhood of the point $(x) = 0$, $(y) = 0$, which

includes the region defined for the transformation (24), this theorem follows at once from Theorem 4, with the exception of the requirement that the branches of the inverse shall be distinct in any point in which they are analytic. That the inverse is defined in all points of a complete neighborhood $\bar{R}$ of $(x) = 0$, follows as in Theorem 2. To see that in general no two branches of this solution are the same at a point of $\bar{R}$, take a series of steps similar to those used in deriving equations (22). Suppose the notation so chosen that this leads to

$$
\begin{aligned}
y_2 &= a_{n-2}(y_1; x_2, \cdots, x_n), \\
y_3 &= a_{n-3}(y_1; x_2, \cdots, x_n), \\
&\cdots\cdots\cdots\cdots \\
y_n &= a_0(y_1, x_2, \cdots, x_n), \\
x_1 &= f_1(y_1, a_{n-2}, \cdots, a_0) = F(x_2, \cdots, x_n; y_1),
\end{aligned} \tag{26}
$$

where

$$
F(0, \cdots, 0) = F_{y_1}(0, \cdots, 0; 0) = \cdots = F_{y_1^{k-1}}(0, \cdots, 0; 0) = 0,
$$

$$
F_{y_1^k}(0, \cdots, 0; 0) \neq 0.
$$

The Weierstrassian preparation theorem puts in evidence the k-fold root y_1 of $F - x_1 = 0$, a continuous function of $x_1, \cdots, x_n$ which satisfies the conditions of Theorem 5. If the Weierstrassian polynomial has a discriminant not identically zero, its root will have k *distinct* values in any point not on the locus defined by equating to zero the discriminant. But this discriminant is equivalent to the result of eliminating y_1 from $F(x_2, \cdots, x_n; y_1) - x_1$ and $F_{y_1} - 0$, and therefore can not be zero for all values of $(x_1, \cdots, x_n)$. For the necessary and sufficient condition that $F - x_1$ and F_{y_1} admit a common factor throughout the region in which they are defined is that the discriminant vanish identically. Since F does not involve x_1, the function $F - x_1$ is irreducible and is not a factor of F_{y_1}. Therefore the discriminant is not identically zero, and except at points for which this discriminant is zero the solution y_1 of $F - x_1 = 0$ has k distinct values; consequently in this point equations (25) have an analytic inverse with k distinct branches. Further, the discriminant, while not identically zero, is always zero at the origin, and hence is zero along a definite locus through the origin. Along this locus the continuous solution found for equations (25) is less than k valued. Since $F - x_1$ is irreducible, y_1 is a single k-valued function of $x_1, \cdots, x_n$, which can not be broken up into two or more functions each with a lower order of multiplicity. Therefore, if $k > 1$, the inverse transformation is defined by functions everywhere continuous, but not everywhere analytic. This completes the proof of Theorem 5.

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ON THE APPROXIMATE REPRESENTATION OF AN INDEFINITE INTEGRAL AND THE DEGREE OF CONVERGENCE OF RELATED FOURIER'S SERIES*

BY

DUNHAM JACKSON

It has been proved by the author† that if $f(x)$ is a function of period 2π which satisfies a Lipschitz condition, it is possible to construct for each positive integral value of n a trigonometric sum $T_n(x)$, of order n or less, such that $|f(x) - T_n(x)|$ remains always inferior to a constant multiple of $1/n$. It will be convenient to use the notation $\mathfrak{T}_n(x)$ as an abbreviation for the words "a trigonometric sum in x of the nth order at most." The symbol $\mathfrak{T}_n$ is accordingly not to be understood as a functional operator. It satisfies, for example, the relations $\mathfrak{T}_n(x) + \mathfrak{T}_n(x) = \mathfrak{T}_n(x)$, $\mathfrak{T}_n(x) + \mathfrak{T}_{n+1}(x) = \mathfrak{T}_{n+1}(x)$. With this convention, we may express the theorem stated above by the equation

$$f(x) = \mathfrak{T}_n(x) + O\left(\frac{1}{n}\right),$$

where $O(1/n)$ stands for a function which never exceeds a constant multiple of $1/n$.

It will be shown below, as the fundamental result of the first part of the paper, that the theorem quoted has the following consequence: If $f(x)$ is expressible in the form

$$f(x) = \mathfrak{T}_n(x) + O(\varphi(n)),$$

then there exists for any indefinite integral of $f(x)$ a representation of the form

$$\int f(x)\,dx = \mathfrak{T}_n(x) + O\left(\frac{\varphi(n)}{n}\right),$$

provided that the obviously necessary condition is satisfied, that $f(x)$ be such as to make the integral periodic.

* Presented to the Society, September 11 and October 26, 1912.

† *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, Dissertation, Göttingen, 1911; pp. 42–46. *On approximation by trigonometric sums and polynomials*, these Transactions, vol. 13 (1912), pp. 491–515. The former paper will be referred to as *Thesis*, the latter as A.

The existence of such a relation is at once suggested by the fact, established by the author* by a modification of the method used to obtain the first result, that if $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition, then

$$f(x) = \mathfrak{T}_n(x) + O\left(\frac{1}{n^k}\right).$$

This generalized proposition is now seen to be a consequence of the simpler one. But we shall obtain more than a new proof of the generalized theorem, namely, a convenient means, not previously recognized, of determining for all values of k numerical limits for the constant multipliers that are involved implicitly in the notation $O(1/n^k)$. Only the cases $k=1$ and $k=2$ had been subjected to detailed investigation in this regard, and the result obtained in the latter of these cases will now be considerably improved, with the aid of a modification of the main theorem.

As a further application of the method a result of still greater generality will be deduced.

The considerations thus far outlined form the substance of the first part of the paper, dealing with approximation by means of trigonometric sums of the nth order. In the second part, the problem of approximation by polynomials of the nth degree will be discussed in the same manner, and corresponding results will be obtained, though with some divergence in the details.

The problems discussed in the third division of the paper, relating mainly to Fourier's series, are at the outset of a somewhat different nature. Their connection with what precedes will be pointed out at the beginning of that section, just before they are taken up in detail.

I. Trigonometric Approximation.

The notation $\mathfrak{T}_n(x)$ has already been explained. We shall introduce also the symbol $\theta(n,x)$ (or $\theta(x)$) to represent "a function of n and x (or of x alone) which never exceeds 1 in absolute value," not as a functional symbol, but as a shorthand expression to be used without alteration to refer to as many different functions of the character indicated as may occur. Every equation in which this symbol appears, in the present section, is understood to hold for all real values of x. With this explanation, we can state the theorem which forms our starting-point as follows:

Theorem I: *There exists an absolute constant K_1 having the following property: If $f(x)$ is a function of the real variable x, of period 2π, which everywhere satisfies the Lipschitz condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

* *Thesis*, Satz VII; A, Theorem III. The proof in A is simpler than that in the thesis.

then, for every positive integral value of n,

$$f(x) = \mathfrak{T}_n(x) + \frac{K_1 \lambda}{n} \theta(n, x).$$

The meaning of the last line is that the function $f(x)$ can be represented by a trigonometric sum of the nth order or lower with an error not exceeding $K_1 \lambda / n$. For the proof of this theorem, the reader is referred to my thesis, pp. 42–46, and the paper A, Theorem I. We assume also the following theorem from A (Theorem VI):

The constant K_1 *may be given the numerical value* 3, *and this is not the smallest admissible value; but a value smaller than* $\pi / 2$ *would be incorrect.*

The fundamental result to be established in this section is

THEOREM II: *If* $f(x)$ *is an integrable* function of period* 2π, *the integral of which over an interval of length* 2π *is zero, so that an indefinite integral of* $f(x)$ *is periodic, and if*

$$f(x) = \mathfrak{T}_n(x) + \epsilon\theta(x),$$

ϵ *being a constant, then*

$$\int f(x)\, dx = \mathfrak{T}_n(x) + \frac{2K_1 \epsilon}{n} \theta(x).$$

The constant K_1 is that of Theorem I. The formulation above has been chosen to emphasize the fact that the theorem may be stated and proved without reference to more than a single value of n. Of course it follows at once that if $f(x) = \mathfrak{T}_n(x) + \varphi(n)\theta(n, x)$, for several or infinitely many or all values of n, then

$$\int f(x)\, dx = \mathfrak{T}_n(x) + \frac{2K_1 \varphi(n)}{n} \theta(n, x),$$

for these values of n.

For the proof of the theorem, set

$$f(x) = T_n(x) + R_n(x),$$

where $T_n(x)$ is a particular trigonometric sum of the nth order at most, and

$$|R_n(x)| \leqq \epsilon.$$

Let a_0 be the constant term of $T_n(x)$. There is no reason why this term should necessarily be zero, as would be the case with the constant term in the Fourier's development of $f(x)$. However, the magnitude of a_0 is not unrestricted; we can be certain that $|a_0| \leqq \epsilon$. For if we write down the relation

$$\int_0^{2\pi} f(x)\, dx = \int_0^{2\pi} [T_n(x) - a_0]\, dx + \int_0^{2\pi} [a_0 + R_n(x)]\, dx,$$

* In the ordinary sense or in the sense of LEBESGUE.

the integral of $f(x)$ is zero by hypothesis, and that of $T_n(x) - a_0$ is zero because the integrand is a trigonometric sum without constant term, and consequently

$$\int_0^{2\pi} [a_0 + R_n(x)]\,dx = 0,$$

from which it appears that $|a_0|$ can not exceed the maximum of $|R_n(x)|$.

It is clear that if the theorem is true for one indefinite integral of $f(x)$ it is true for all, since two indefinite integrals of the same function differ only by a trigonometric sum of order zero. Let us consider the integral

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x [T_n(x) - a_0]\,dx + \int_0^x [a_0 + R_n(x)]\,dx.$$

The first of these two terms is a trigonometric sum of the same order as $T_n(x)$. The second is a continuous function of x satisfying a Lipschitz condition with coefficient 2ϵ. For

$$\left|\int_0^{x_2} [a_0 + R_n(x)]\,dx - \int_0^{x_1} [a_0 + R_n(x)]\,dx\right| = \left|\int_{x_1}^{x_2} [a_0 + R_n(x)]\,dx\right|$$

$$\leqq \left|\int_{x_1}^{x_2} (\epsilon + \epsilon)\,dx\right| = 2\epsilon\,|x_2 - x_1|.$$

Hence the term has the form $\mathfrak{T}_n(x) + \theta(x) \cdot 2K_1\epsilon/n$, by Theorem I, and $F(x)$ has the same form, as was asserted in the theorem to be proved.

It may be observed that if it had been known through any specialization of the problem that a_0 was zero, the factor 2 would not have entered, and we should have had an error not exceeding $K_1\epsilon/n$.

If we suppose that $f(x)$ has a $(k-1)$th derivative satisfying the Lipschitz condition

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda\,|x_2 - x_1|,$$

we may apply Theorem I to the representation of $f^{(k-1)}(x)$ by a trigonometric sum, and then Theorem II to the representation of $f^{(k-2)}(x), \cdots, f'(x), f(x)$, successively,* and deduce the theorem that $f(x)$ has the form $\mathfrak{T}_n(x) + \theta(n, x) \cdot 2^{k-1}K_1^k\lambda/n^k$. Thus Theorem III of the article A, which differs from this in having a new undetermined constant K_k for each value of k instead of the expression $2^{k-1}K_1^k$, is proved more simply than

* Of course the condition that the integral over an interval of length 2π shall be zero is not imposed on $f(x)$ here, but concerns only the derivatives, by which it is satisfied automatically.

before, and at the same time rendered considerably more definite. If we give K_1 the value 3, which we know to be admissible, we have at once numerical values for all the constants K_k.

It is not difficult, however, to improve materially upon the values obtained in this way, namely, to justify the omission of the factor 2^{k-1} and the substitution of 3^k for K_k. Up to the present we have used Theorem I as it was stated, without reference to the method by which it was proved. If it should be shown by any method whatever that Theorem I remains true with a smaller value of K_1, then this value could be substituted for K_1 throughout the preceding work, and everything that has been said would still be true. We have not derived any benefit from the fact that Theorem I was proved (in the article A, as also in the thesis) by constructing trigonometric sums of a special form which furnished the degree of approximation desired. That fact can however be made very useful.

It has been remarked that if the given approximating function in the hypothesis of Theorem II lacks the constant term, the factor 2 in the conclusion can be dispensed with. The trigonometric sums used to prove Theorem I were of the form

$$C\int_{-\pi}^{\pi} f(v)\left[\frac{\sin \frac{1}{2}m(v-x)}{m\sin\frac{1}{2}(v-x)}\right]^4 dv,$$

where C is a constant and m is a positive integer. It was shown that the fourth power in the integrand is a trigonometric sum in $v-x$, so that, on expanding the several terms and integrating, a trigonometric sum in x is obtained. And now it is clear that if the integral of $f(x)$ from $-\pi$ to π is zero, the constant term in the expression above, regarded as a trigonometric sum in x, will vanish.

In fact, any terms will be lacking which may be lacking in the Fourier's expansion* of $f(x)$. For the fourth power involved, being an even function of $v-x$, and being at the same time a trigonometric sum in this argument, is merely a sum of cosines of multiples of $v-x$. The terms in $\cos px$ and $\sin px$ in the sum obtained on integrating are of the forms

$$C\int_{-\pi}^{\pi} f(v)\cos pv\cos px\,dv, \qquad C\int_{-\pi}^{\pi} f(v)\sin pv\sin px\,dv,$$

and will vanish if the corresponding Fourier's coefficients do so. In fact, it will be seen upon closer inspection that our method amounts to a particular kind of "summation" of the Fourier's series, as has frequently been pointed out in connection with similar methods before; that is, our trigonometric

* In this connection, cf. BÔCHER, *Introduction to the theory of Fourier's series*, Annals of Mathematics, second series, vol. 7 (1906), pp. 81–152 (also published separately), pp. 102–103, and the first footnote on the latter page.

sums may be regarded as obtained from the partial sums of corresponding order in the Fourier's series for $f(x)$ by multiplying the terms of the latter by constant factors, independent of the function developed, which approach unity, in the case of any particular term, as the number of terms is increased. We are in a position to state that *if $f(x)$ satisfies a Lipschitz condition with coefficient λ, then $f(x)$ can be represented, with an error not exceeding $3\lambda/n$, by a trigonometric sum of the nth order at most, containing no terms which do not occur in the Fourier's development of $f(x)$*. Of course nothing that has been said would justify the conclusion that a smaller value of K_1 in Theorem I, which might be obtained by a different method, would be applicable here. On the other hand, any value obtained by a further refinement of the same method would be admissible.

We are concerned here only with the vanishing of the constant term. Let it be assumed that $f(x)$ is periodic, with period 2π, and has a derivative of order $(k-1)$ which satisfies a Lipschitz condition with coefficient λ. Then $f^{(k-1)}(x) = \mathfrak{T}_n(x) + \theta(n, x) \cdot 3\lambda/n$. If $k = 1$, nothing more is desired. If $k > 1$, then $f^{(k-1)}(x)$ has an integral which is periodic, and the constant term in the sum $\mathfrak{T}_n(x)$ above may be assumed to be zero. Consequently $f^{(k-2)}(x) = \mathfrak{T}_n x) + \theta(n, x) \cdot 3^2\lambda/n^2$, by the version of Theorem II which is applicable in such cases. If $k = 2$, we stop at this point; we have $K_2 = 9$, instead of the much cruder value 20 (or 18.2) of the earlier paper A (Theorem VIII, with the text immediately preceding).

If $k > 2$, our next step is to show that in the trigonometric sum representing $f^{(k-2)}(x)$ the constant term may again be assumed to vanish. To this end, we go back to the representation of $f^{(k-1)}(x)$. We may set

$$f^{(k-1)}(x) = t_n(x) + r_n(x),$$

where $t_n(x)$ is a trigonometric sum of order n or less with no constant term, and $|r_n(x)| \leqq 3\lambda/n$. By integrating, we get

$$f^{(k-2)}(x) = C + \int_0^x t_n(x)\,dx + \int_0^x r_n(x)\,dx,$$

where C is a constant. The first integral on the right is a trigonometric sum, since $t_n(x)$ lacks the constant term. The second integral is a continuous function of x satisfying a Lipschitz condition with coefficient $3\lambda/n$, and will be represented by a trigonometric sum with a corresponding degree of approximation, as was done in the proof of Theorem II; but we are not quite ready for this step. Let us set

$$C + \int_0^x t_n(x)\,dx = T_n(x) = \mathfrak{T}_n(x).$$

Then

$$f^{(k-2)}(x) = T_n(x) + \int_0^x r_n(t)\,dt.$$

Denoting the constant term in $T_n(x)$ by c_0, and subtracting and adding this term, we have

$$f^{(k-2)}(x) = T_n(x) - c_0 + c_0 + \int_0^x r_n(t)\,dt.$$

Since $T_n(x) - c_0$ is a trigonometric sum without constant term,

$$\int_0^{2\pi} [T_n(x) - c_0]\,dx = 0.$$

Furthermore,

$$\int_0^{2\pi} f^{(k-2)}(x)\,dx = f^{(k-3)}(2\pi) - f^{(k-3)}(0);$$

and this is zero, since we suppose that $k \overline{\overline{>}} 3$, and hence $f^{(k-3)}(x)$ is really a derivative of $f(x)$ or else $f(x)$ itself, and so periodic. It follows that

$$\int_0^{2\pi} \left\{ c_0 + \int_0^x r_n(t)\,dt \right\} dx = 0.$$

This is the essential point of the present argument. It shows that the expression

$$c_0 + \int_0^x r_n(t)\,dt,$$

which satisfies the Lipschitz condition with coefficient $3\lambda / n$, can be represented by a trigonometric sum of order n at most, *with no constant term*, so that the error shall be not greater than $3^2\lambda / n^2$. Consequently $f^{(k-2)}(x)$ has a representation of the same sort.

Thus we are enabled to apply Theorem II once more in its specialized form, obtaining $f^{(k-3)}(x) = \mathfrak{T}_n(x) + \theta(n, x) \cdot 3^3\lambda / n^3$. The reasoning of the preceding paragraph, *mutatis mutandis*, may be repeated step by step to show that here also the constant term in the trigonometric sum may be assumed to vanish, if $k \overline{\overline{>}} 4$. And so we proceed until we get a representation of $f'(x)$ lacking the constant term, and a representation of $f(x)$, in which the presence of a constant term is immaterial, the error in the last representation being not greater than $3^k\lambda / n^k$. We have thus the remarkably simple result, which we will state at length, without abbreviation:

THEOREM III: *If $f(x)$ is a function of period 2π possessing a $(k-1)$th derivative which everywhere satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \overline{\overline{<}} \lambda\,|x_2 - x_1|,$$

then there exists for each positive integral value of n a trigonometric sum $T_n(x)$, of the nth order at most, such that for all values of x

$$|f(x) - T_n(x)| \leqq \frac{3^k \lambda}{n^k}.$$

The bearing of this result on the theory of the convergence of Fourier's series is indicated in Theorem V of the article A. We shall do better, however, to use another method (Theorem X below).

To prepare the way for a more general theorem, suppose that $f(x)$ is any continuous function of x with the period 2π. Corresponding to this function $f(x)$, let a function $\omega(\delta)$ of the positive variable δ be defined, so that, however x_1 and x_2 may be chosen subject to the condition $|x_2 - x_1| \leqq \delta$, we shall always have $|f(x_2) - f(x_1)| \leqq \omega(\delta)$. For our present purpose no further restriction is laid on the function ω. It follows from its definition that $\omega(\delta)$ never decreases when δ increases. The author has shown* that under these circumstances

$$f(x) = \mathfrak{T}_n(x) + \left(\frac{K_1}{2\pi} + 2\right)\omega\left(\frac{2\pi}{n}\right)\theta(n, x),$$

where K_1 is the constant of Theorem I and the relation holds for all positive integral values of n. The proof† is so simple that it may be reproduced here. We first define a function $f_1(x)$ equal to $f(x)$ at the points $x = 2s\pi/n$ for all integral values of s, and linear in each of the intervals of length $2\pi/n$ between two successive points of this set. Then,‡ on the one hand,

$$|f(x) - f_1(x)| \leqq 2\omega\left(\frac{2\pi}{n}\right)$$

for all values of x; and, on the other hand, $f_1(x)$ satisfies the conditions for the application of Theorem I, with

$$\lambda = \omega\left(\frac{2\pi}{n}\right) \Big/ \frac{2\pi}{n}.$$

Hence the proposition follows at once.

By starting from this theorem, and applying Theorem II repeated a sufficient number of times, as we have done before, we deduce the following general theorem:

* *Thesis*, p. 48.

† It is readily seen that nothing need be changed if $f(x)$ is supposed to be discontinuous, but to remain finite.

‡ The central idea of this proof is taken from LEBESGUE, who uses it in a different connection; Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; p. 202 of the volume, p. 19 of the article.

THEOREM IV: *If $f(x)$ has the period 2π and possesses a continuous kth derivative satisfying the condition that*

$$|f^{(k)}(x_2) - f^{(k)}(x_1)| \leqq \omega(\delta)$$

whenever $|x_2 - x_1| \leqq \delta$, *then*

$$f(x) = \mathfrak{T}_n(x) + O\left(\frac{1}{n^k}\omega\left(\frac{2\pi}{n}\right)\right),$$

uniformly for all values of x.

By saying that this relation holds "uniformly" we mean, naturally, that the constant multipliers implied in the O-notation are independent of x. Of course numerical values could be calculated for them without difficulty.

The result just obtained can be applied to the theory of Fourier's series in the same manner as Theorem III above.*

II. POLYNOMIAL APPROXIMATION.

Corresponding to the notation $\mathfrak{T}_n(x)$ of the first part of the paper, we shall introduce the symbol $\mathfrak{P}_n(x)$ to replace the words "a polynomial in x of the nth degree at most." The signs $\theta(x)$, $\theta(n, x)$, will be used as before, except that relations involving them are understood to hold, no longer for all real values of x, but for all values of x in the closed interval under consideration.

We set out from the following theorem, which we shall quote without proof:

THEOREM V: *There exists an absolute constant L_1 having the following property:*† *If $f(x)$ is a function which satisfies the Lipschitz condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

throughout the closed interval $a \leqq x \leqq b$ of length l, then, for every positive integral value of n,

$$f(x) = \mathfrak{P}_n(x) + \frac{L_1 l \lambda}{n}\theta(n, x),$$

throughout the interval. Furthermore,‡ *the constant L_1 may be given the value $\frac{1}{2}K_1$, if K_1 is any number which is admissible in Theorem I; so that, in particular,*§ *L_1 may have the value $\frac{3}{2}$; but it can not have a value smaller than $\frac{1}{2}$.*

From this theorem we obtain readily

* The theorem deduced in this way is given, for a large class of cases, on p. 92 of a recent essay of BERNSTEIN to which more extended reference will be made in a later footnote.

† *Thesis*, p. 39; A, Theorem II.

‡ A, Theorem II.

§ A, Theorem VII. There is a misprint in the statement of the theorem, and also four lines above; the symbol L_2 should be L_1.

THEOREM VI: *If, in the interval* $a \leqq x \leqq b$ *of length* l, *the function* $f(x)$ *is integrable,* and*

$$f(x) = \mathfrak{P}_n(x) + \epsilon\,\theta(x),$$

ϵ *being a constant, then*

$$\int f(x)\,dx = \mathfrak{P}_{n+1}(x) + \frac{L_1 l \epsilon}{n+1}\,\theta(x).$$

According to the hypothesis concerning $f(x)$, we may write

$$f(x) = \Pi_n(x) + R_n(x),$$

where $\Pi_n(x)$ is a polynomial of the nth degree at most, and

$$|R_n(x)| \leqq \epsilon.$$

Consider the integral

$$F(x) = \int_a^x f(x)\,dx = \int_a^x \Pi_n(x)\,dx + \int_a^x R_n(x)\,dx.$$

The first term here is a polynomial of degree $n+1$ at most. The second is a function satisfying the condition

$$\left| \int_a^{x_2} R_n(x)\,dx - \int_a^{x_1} R_n(x)\,dx \right| \leqq \epsilon\,|x_2 - x_1|;$$

so that by Theorem V, with n replaced by $n+1$, it has the form $\mathfrak{P}_{n+1}(x) + \theta(x) \cdot L_1 l \epsilon / (n+1)$. It follows that $F(x)$ has the same form.

Then a moment's reflection yields the theorem:

THEOREM VII: *If* $f(x)$ *is a function possessing a* $(k-1)$*th derivative which satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda\,|x_2 - x_1|$$

throughout the interval $a \leqq x \leqq b$ *of length* l, *then, for each integral value of* $n \geqq k$,

$$f(x) = \mathfrak{P}_n(x) + \frac{L_1^k l^k \lambda}{n(n-1)\cdots(n-k+1)}\,\theta(n, x).$$

First, we express $f^{(k-1)}(x)$ in the form

$$f^{(k-1)}(x) = \mathfrak{P}_{n-k+1}(x) + \theta(x) \cdot L_1 l \lambda / (n-k+1),$$

by Theorem V; and then we apply $k-1$ times in succession the theorem just obtained, to arrive at the conclusion stated.

This recalls at once Theorem IV of the paper A, which attaches to the same hypothesis the conclusion that, for $n \geqq k-1$, the function $f(x)$ has the

* In the ordinary sense or in the sense of LEBESGUE.

form $f(x) = \mathfrak{P}_n(x) + \theta(n, x) \cdot L_k l^k \lambda / n^k$, where L_k is a constant dependent only on k. If we leave out of account for the moment the fact that one of these results is stated for all values of n from $n = k - 1$, the other only for values from $n = k$ on, we see that we have a new and simple proof of the old theorem, since

$$\frac{L_1^k l^k \lambda}{n(n-1)\cdots(n-k+1)} = \frac{n^k}{n(n-1)\cdots(n-k+1)} \cdot \frac{L_1^k l^k \lambda}{n^k} \lessgtr \frac{k^k}{k!} \cdot \frac{L_1^k l^k \lambda}{n^k} \qquad (n \geqq k).$$

Further, we have a means of computing numerical values for the constants L_k which are correct if we suppose that we are concerned only with values of $n \lessgtr k$. When numerical results are desired, however, it is manifestly better to use the formally less simple statement of our present Theorem VII as it stands.

As for the representation of $f(x)$ by a polynomial of degree $k - 1$, it is obvious that $f^{(k-1)}(x)$ can be represented by a constant, namely, by its value at the middle point of the interval, with an error not greater than $\frac{1}{2} l\lambda$; and it follows from this, by repeated application of Theorem VI, that $f(x)$ admits the representation $f(x) = \mathfrak{P}_{k-1}(x) + \theta(x) \cdot \frac{1}{2} L_1^{k-1} l^k \lambda / (k-1)!$. As it is certain* that L_1 cannot be less than $\frac{1}{2}$, we may reason further that

$$\frac{L_1^{k-1} l^k \lambda}{2(k-1)!} \lessgtr \frac{L_1^k l^k \lambda}{(k-1)!} = \frac{(k-1)^k}{(k-1)!} \cdot \frac{L_1^k l^k \lambda}{(k-1)^k},$$

which is an expression of precisely the form demanded by the old theorem. If we multiply L_1^k by the larger of the quantities $k^k / k!$ and $(k-1)^k / (k-1)!$, we have a value of L_k which is applicable in that theorem without exception. For $k = 2$, the value $L_1 = \frac{3}{2}$ gives $L_2 = 4\frac{1}{2}$, which is better than the value somewhat less than 10 that was previously known.† It would be easy by means of special considerations to reduce this value still further, but we shall not undertake to do so here.

The present method throws a new light on the difference between Theorems III and IV of A, one holding for $n \lessgtr 1$ and the other only for $n \lessgtr k - 1$. This difference appears now as a consequence of the fact that the integral of a trigonometric sum (without constant term) is a trigonometric sum of the same order, while the integral of a polynomial is a polynomial of higher degree.

Lastly, we seek an analogon of Theorem IV for the case of polynomial approximation. It is known‡ that if $|f(x_2) - f(x_1)| \lessgtr \omega(\delta)$ whenever $|x_2 - x_1| \lessgtr \delta$ and x_1, x_2 are in the interval $a \lessgtr x \lessgtr b$ of length l, then $f(x) = \mathfrak{P}_n(x) + \theta(x) \cdot (L_1 + 2)\, \omega(l/n)$ for all positive integral values

* Cf. the last clause of Theorem V above.

† A, Theorem IX.

‡ *Thesis*, pp. 40–41.

of n. In this case, the interval from a to b is divided into n equal parts, and a broken-line function $f_1(x)$ is constructed which is equal to $f(x)$ at the points of division; the proof need not be further recalled.

Combining this fact with Theorem VI, in a manner already sufficiently illustrated, we find at once the generalization:

THEOREM VIII: *If $f(x)$ possesses a continuous kth derivative satisfying the condition that*

$$|f^{(k)}(x_2) - f^{(k)}(x_1)| \leqq \omega(\delta),$$

whenever $|x_2 - x_1| \leqq \delta$ and x_1, x_2 are in the interval $a \leqq x \leqq b$ of length l, then

$$f(x) = \mathfrak{P}_n(x) + O\left(\frac{1}{n^k}\omega\left(\frac{l}{n-k}\right)\right)$$

uniformly for all values of x in the interval.

The constant multiplier implied in the O-symbol depends only on k and l; as far as l is concerned, it may be taken proportional to l^k.

The reader is referred to the first two theorems of a recent paper by the author,* with the accompanying discussion, for an indication of the manner in which the above result may be applied in the theory of Legendre's series.

III. FOURIER'S SERIES.

We shall be concerned in this section principally with the approximation afforded by trigonometric sums chosen in a particular way, namely, those obtained by breaking off a Fourier's series after the terms of the nth order. In particular we shall investigate the degree of convergence of the series obtained by integrating or differentiating a given Fourier's series a given number of times. If the number of integrations or differentiations performed is even, the problem is very simple, and does not call for methods closely related to those of the earlier parts of the paper.† But if the number is odd, the direct method is no longer sufficient, and Theorem II of the first part is adduced to furnish the needed connection. It is this circumstance which unites the present section with what goes before.‡

* These Transactions, vol. 13 (1912), pp. 305–318.

† This case was treated to some extent in my thesis, pp. 82–86, as far as differentiation is concerned.

‡ Since this paper was written, the author has received, through the kindness of S. BERNSTEIN, a copy of the latter's Mémoire Couronné, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Bruxelles, 1912, in which methods are developed that render possible a more satisfactory discussion of the problems of the present section. The reader of both papers will have little difficulty in seeing how some of the theorems below may be improved in statement. BERNSTEIN is more especially interested, to be sure, in the problem of polynomial approximation, but he provides the instruments for the treatment of the trigonometric case, in particular (p. 20) the theorem:

"S étant une suite trigonométrique quelconque, si son module reste inférieur à L, le module de sa dérivée reste inférieur à $2nL$."

It is a well-known fact, of which use will be made more than once below, that if a function can be represented to a known degree of approximation by a trigonometric sum of specified order, an upper limit for the error of the corresponding partial sum of its Fourier's series can be deduced. With this in mind, it may be seen how the theorems concerning the differentiation of Fourier's series throw light on the degree of approximation by a trigonometric sum of given order attainable in the case of a derivative of a function which can itself be approximately represented by such a sum with known accuracy. A further connection with the first part thus becomes apparent. The approximation to the derivative obtained in this way is not of such an order as to yield a converse of the integration-theorem of Part I, but this is not surprising if we consider the special nature of the trigonometric sums employed.*

The other results of the section are, in the first place, the theorem concerning the degree of convergence of the Fourier's series for a function satisfying certain conditions (Theorem X), with a consequence readily deduced from it (Theorem XI); further, the theorem concerning the convergence of a series associated in a particularly simple way with a given Fourier's series (Theorem XIII); and the concluding theorems,† which give *necessary* conditions for the possibility of certain degrees of approximation.

To begin the detailed investigation, let us suppose that the series

$$(1) \qquad (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

converges uniformly for all values of x to the value‡ $f(x)$. Let us set

$$r_n(x) = \sum_{h=n+1}^{\infty} (a_h \cos hx + b_h \sin hx).$$

Let $\varphi(n)$ be a function of n such that

$$| r_n(x) | \overline{\leqq} \varphi(n)$$

for all values of x, while $\varphi(n+1) \overline{\leqq} \varphi(n)$ for each value of n and $\lim_{n=\infty} \varphi(n) = 0$. Such a function will surely exist; for example, $\varphi(n)$ may be defined as the maximum of $| r_h(x) |$ for $h \overline{\geqq} n$. The function $\varphi(n)$, then, measures the rate of convergence of the series (1), in the sense that it gives an upper limit for the magnitude of the remainder. We shall obtain a function performing the same service with reference to certain other series related to (1).

If q is any positive integer, we have, as an immediate consequence of our hypotheses,

$$| r_{n+q}(x) | \overline{\leqq} \varphi(n+q) \overline{\leqq} \varphi(n),$$

* Cf. BERNSTEIN, loc. cit., p. 27.

† These are only partly new; see references given in connection with them below.

‡ It must accordingly be the Fourier's series for $f(x)$.

and hence

$$(2) \qquad \left| \sum_{h=n+1}^{n+q} (a_h \cos hx + b_h \sin hx) \right| = | r_n(x) - r_{n+q}(x) | \leqq 2\varphi(n).$$

Now let us set

$$\rho_{np}(x) = \sum_{h=n+1}^{n+p} \frac{1}{h^l} (a_h \cos hx + b_h \sin hx),$$

where l is a positive integer. Remembering that (2) holds for all the values $q = 1, \cdots, p$, we see that we have a sum to which Abel's lemma is applicable; we conclude that

$$| \rho_{np}(x) | \leqq \frac{2\varphi(n)}{(n+1)^l}.$$

As the right-hand member here is independent of p, and approaches zero as n becomes infinite, the series

$$(3) \qquad (a_1 \cos x + b_1 \sin x) + \frac{1}{2^l} (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

converges uniformly, and if the remainder of this series after the terms in nx is set equal to $\rho_n(x)$, we have

$$| \rho_n(x) | \leqq \frac{2\varphi(n)}{(n+1)^l}.$$

Now if $l = 2l'$ is an even number, the series (3), multiplied by $(-1)^{l'}$, is the Fourier's series of a function obtained by integrating (1) repeatedly l times.* We may state our conclusion in the following form:

THEOREM IXa: *If the remainder after the term of the nth order in the Fourier's series for a function $f(x)$ never exceeds $\varphi(n)$ in absolute value, where $\varphi(n)$ is a function of the nature specified above, then the corresponding remainder in the Fourier's series for a periodic lth integral of $f(x)$, in the case that l is even, does not exceed $2\varphi(n)/(n+1)^l$ in absolute value.*

It is readily seen how this statement has to be modified so as to be applicable to an integral which is not periodic, because of the presence of polynomial terms; this case would surely arise if the Fourier's series for $f(x)$ had a constant term different from zero.

The method just used gives no direct information about the series obtained by integrating (1) an odd number of times. To make progress here, we apply our Theorem II, modified in the manner indicated at the close of its proof. The partial sum of (1) to terms of the nth order is a trigonometric sum without constant term which represents $f(x)$ with an error less than or equal to $\varphi(n)$;

* Cf., e. g., BÔCHER, loc. cit., p. 117. The proof can be made particularly simple here, since all the series concerned are uniformly convergent.

it follows that $\int f(x)\,dx$ can be represented by a trigonometric sum of the nth order with an error not exceeding $3\,\varphi(n)/n$. From this we deduce the further consequence* that the remainder after terms of the nth order, $n \geqq 1$, in the Fourier's series for $\int f(x)\,dx$ does not exceed $[3K\varphi(n)\log n]/n$, where K is a certain absolute constant; if we restrict n to values not less than 5, we may say more specifically that the remainder does not exceed $[6\varphi(n)\log n]/n$. From this point we may apply our earlier reasoning, in fact, precisely the theorem IXa; we find the supplementary result:

THEOREM IXb: *If the number l of Theorem IXa is odd instead of even, the remainder in the Fourier's series for the lth integral does not exceed*

$$\frac{12\varphi(n)\log n}{n(n+1)^{l-1}}$$

in absolute value, provided $n \geqq 5$.

More roughly, the expressions $2\varphi(n)/n^l$ and $[12\varphi(n)\log n]/n^l$ may be assigned as limits of error in the two cases.

To make use of Theorem IXa in another way, let $f(x)$ be a function which fulfils the hypotheses of Theorem III; it is not our purpose, however, to apply the latter theorem to $f(x)$. If the number k which figures in the hypothesis is odd, so that $k-1$ is even, we say at first merely that $f^{(k-1)}(x)$ has the form $\mathfrak{T}_n(x) + \theta(n, x)\cdot 3\lambda/n$, by Theorem I. Hence we infer that $f^{(k-1)}(x)$ is represented by its Fourier sum of the nth order, if we may use this abbreviation for the partial sum of its Fourier's series, with an error never greater than $(6\lambda\log n)/n$ for $n \geqq 5$. By Theorem IXa we pass to $f(x)$ itself, and see that the latter function is represented by its Fourier sum of the nth order with an error not exceeding $(12\lambda\log n)/n^k$. If k is even, and $k-1$ odd, we begin by expressing $f^{(k-2)}(x)$ as $\mathfrak{T}_n(x) + \theta(n, x)\cdot 9\lambda/n^2$, by Theorem III; for the Fourier's series, we get the limits of error $(18\lambda\log n)/n^2$, for $f^{(k-2)}(x)$, and $(36\lambda\log n)/n^k$, for $f(x)$. In other words, we have†

THEOREM X: *If $f(x)$ is a function of period 2π possessing a $(k-1)$th derivative which everywhere satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda\,|x_2 - x_1|,$$

then $f(x)$ is everywhere approximately represented by the partial sum of its Fourier's series to terms of the nth order, $n \geqq 5$, with an error not exceeding $(36\lambda\log n)/n^k$. If k is odd, the factor 36 may be replaced by 12.

By applying Theorem III directly to $f(x)$, and passing then to the Fourier's series, we should have found instead of the number 36 a coefficient depending on k, though not on anything else.

* For proof and further references, see the paragraphs immediately preceding Theorems V and X of article A.

† Cf. BERNSTEIN, loc. cit., pp. 92–93, for a broader theorem, without determination of the numerical factor.

A further consequence of the theorem just obtained is perhaps not without interest. It is readily seen that if $t(x)$ is a finite trigonometric sum of the pth order,

$$t(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_p \cos px + b_p \sin px,$$

where a_p and b_p are not both zero, and if two positive constants m and G exist so that

$$| t^{(k)}(x) | \leqq Gm^k$$

for all values of x and for all, or for infinitely many, values of k, then $p \leqq m$. For we have

$$t^{(k)}(x) = (-1)^k [(a_1 \cos x + b_1 \sin x) + \cdots + p^k (a_p \cos px + b_p \sin px)]$$

or

$$t^{(k)}(x) = (-1)^{\frac{k+1}{2}} [(a_1 \sin x - b_1 \cos x) + \cdots + p^k (a_p \sin px - b_p \cos px)]$$

according as k is even or odd. In each of these expressions let us set $x = x_0$, where x_0 is a value such that neither $a_p \cos px_0 + b_p \sin px_0$ nor $a_p \sin px_0 - b_p \cos px_0$ is zero, a value which will surely exist, and divide by m^k. If $p > m$, each quotient becomes infinite* with k, which is contrary to hypothesis.

Now let $f(x)$ be any function of period 2π having derivatives of all orders which satisfy the condition imposed on the derivatives of $t(x)$ above. If k is one of the indices for which $| f^{(k)}(x) | \leqq Gm^k$, it follows from the law of the mean that

$$| f^{(k-1)}(x_2) - f^{(k-1)}(x_1) | \leqq Gm^k | x_2 - x_1 |,$$

whatever the values of x_1 and x_2 may be. By Theorem X, the function $f(x)$ differs from its Fourier sum of the nth order, $n > 4$, by not more than $(36Gm^k \log n)/n^k$. If we give n a fixed value greater than m, this relation, holding for infinitely many values of k and so for values of k as large as we please, requires that $f(x)$ be identically equal to the Fourier sum. Being a finite trigonometric sum, $f(x)$ can not be of order higher than m, and we have

THEOREM XI: *If $f(x)$ is a function of period 2π possessing derivatives of all orders, and if, for infinitely many values of k,*

$$| f^{(k)}(x) | \leqq Gm^k$$

for all values of x, where m and G are constants, then $f(x)$ is a finite trigonometric sum of order not higher than m.

It may be observed that there has been no occasion to exclude the possibility that $0 < m < 1$; in this case, $f(x)$ reduces to a constant.

* This will perhaps be more obvious in case $p > m + 1$ if we divide, not by m^k, but by $(p-1)^k$.

The last paragraphs have been of the nature of a digression. To return to our earlier line of reasoning, suppose again that the series (1) converges uniformly so that $|r_n(x)| \leqq \varphi(n)$ for all values of n and x, but suppose now that φ is a function decreasing so rapidly as n increases that $n^{l+1}\varphi(n)$ decreases monotonically, that is, that $(n+1)^{l+1}\varphi(n+1) \leqq n^{l+1}\varphi(n)$ for every value of n. By l we mean again a positive integer. We set

$$\rho_{np}(x) = \sum_{h=n+1}^{n+p} h^l (a_h \cos hx + b_h \sin hx).$$

We see that this may be rewritten

$$\rho_{np}(x) = \sum_{h=n+1}^{n+p} h^l [r_{h-1}(x) - r_h(x)]$$

$$= \sum_{h=n+1}^{n+p} [(h+1)^l - h^l] r_h(x) + (n+1)^l r_n(x) - (n+p+1)^l r_{n+p}(x),$$

the last expression being derived from the preceding by "partial summation."* In the expression under the sign of summation, we notice that

$$(h+1)^l - h^l = l\xi^{l-1},$$

where ξ is a value between h and $h+1$, and accordingly

$$(h+1)^l - h^l < l(h+1)^{l-1}.$$

Applying this inequality and the fact that $(h+1)/h$ and $h^{l+1}\varphi(h)$ are monotonically decreasing functions, we see that

$$\left| \sum_{h=n+1}^{n+p} [(h+1)^l - h^l] r_h(x) \right| \leqq \sum_{h=n+1}^{n+p} l(h+1)^{l-1}\varphi(h)$$

$$= \sum_{h=n+1}^{n+p} l\left(\frac{h+1}{h}\right)^{l-1} h^{l+1}\varphi(h) \cdot \frac{1}{h^2} \leqq l\left(\frac{n+1}{n}\right)^{l-1} n^{l+1}\varphi(n) \sum_{h=n+1}^{n+p} \frac{1}{h^2}$$

$$\leqq l\left(\frac{n+1}{n}\right)^{l-1} n^{l+1}\varphi(n) \int_n^\infty \frac{dt}{t^2} = l\left(\frac{n+1}{n}\right)^{l-1} n^l \varphi(n).$$

For the terms outside the sign of summation in the expression for $\rho_{np}(x)$, we have the inequalities

$$|(n+1)^l r_n(x)| \leqq (n+1)^l \varphi(n) = \left(\frac{n+1}{n}\right)^l n^l \varphi(n),$$

$$|(n+p+1)^l r_{n+p}(x)| \leqq \left(\frac{n+p+1}{n+p}\right)^l (n+p)^l \varphi(n+p)$$

$$\leqq \left(\frac{n+1}{n}\right)^l n^l \varphi(n),$$

* For a general formulation of this device, cf. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 84.

the last inequality resulting from the fact that, if $n^{l+1}\varphi(n)$ decreases monotonically, then $n^l\varphi(n)$ has the same property.* Hence

$$|\rho_{np}(x)| \leqq l\left(\frac{n+1}{n}\right)^{l-1} n^l\varphi(n) + 2\left(\frac{n+1}{n}\right)^l n^l\varphi(n) \leqq c_l n^l\varphi(n),$$

where c_l is a constant depending only on l. The last expression is independent of p. Since $n^{l+1}\varphi(n)$ decreases monotonically, it must *a fortiori* remain finite as n becomes infinite, and $n^l\varphi(n)$ must approach zero. It follows that the series

$$\sum_{n=1}^{\infty} n^l(a_n \cos nx + b_n \sin nx) \tag{4}$$

converges uniformly. Also, of course, $\rho_{np}(x)$ approaches uniformly the limit

$$\rho_n(x) = \sum_{h=n+1}^{\infty} h^l(a_h \cos hx + b_h \sin hx)$$

as p becomes infinite. Passing to the limit in the last inequality for $|\rho_{np}|$, we find that

$$|\rho_n(x)| \leqq c_l n^l\varphi(n).$$

If l is even, the series (4) is, except perhaps as to sign, the Fourier's series of a continuous lth derivative of $f(x)$, as we shall again call the sum of (1). For a series of this form, when uniformly convergent, is always the Fourier's series of the function which it represents, and we can get (1) from (4) by repeated term-by-term integration. We have proved†

THEOREM XII*a*: *If the absolute value of the remainder after terms of the nth order in the Fourier's series for a function‡ $f(x)$ never exceeds $\varphi(n)$, where φ is such that $n^{l+1}\varphi(n)$ decreases monotonically, then the corresponding remainder in the Fourier's series for the lth derivative of $f(x)$, which will surely exist, does not exceed $c_l n^l\varphi(n)$ in absolute value in the case that l is even, where c_l is a constant depending only on l.*

If, in the hypothesis, we had assumed somewhat less, namely, only that $n^l\varphi(n)\log^2 n$ decreases monotonically, we should still have been able to deduce the uniform convergence of (4) with a remainder, that does not exceed $n^l\varphi(n)\log n$ multiplied by a constant depending on l. It is clearly possible to go further in the same direction.

Let us see what happens if l in the hypothesis of Theorem XII*a* is odd. The integral $\int f(x)\,dx$, where f has been modified, if necessary, by the subtrac-

* For $(n+1)^l\varphi(n+1) = \frac{1}{n+1}(n+1)^{l+1}\varphi(n+1) \leqq \frac{1}{n}n^{l+1}\varphi(n) = n^l\varphi(n)$.

† In connection with Theorems XII*a* and XII*b*, cf. BERNSTEIN, loc. cit., pp. 22–28.

‡ The fact that the series (1) with which we have been working has no constant term is clearly immaterial.

tion of a constant so that the integral shall be periodic, can be represented by a trigonometric sum of the nth order at most with an error never greater than $3\varphi(n)/n$, and is represented by its Fourier sum of the nth order with an error not exceeding* $[3K\varphi(n)\log n]/n$ for $n \geqq 2$. We see that we shall want to modify the hypothesis still further to the extent of supposing that $n^{l+1}\varphi(n)\log n$ decreases monotonically; then we can apply Theorem XII*a*, with l replaced by $l+1$, to $\int f(x)\,dx$, and obtain

THEOREM XII*b*: *If the number l of Theorem XII*a *is odd instead of even, and if φ is such that $n^{l+1}\varphi(n)\log n$ decreases monotonically, then the remainder in the Fourier's series for $f^{(l)}(x)$ does not exceed $C_l n^l \varphi(n)\log n$ in absolute value, when $n \geqq 2$; here C_l is a constant depending only on l.*

In this connection, we observe that from a knowledge of the degree of convergence of the series

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{5}$$

information can be derived as to the convergence of a certain series which is neither its integral nor its derivative, namely the series

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx). \tag{6}$$

Particularly interesting cases are those in which all the a's or all the b's are zero. If the first series merely converges uniformly, the second may diverge for some values of the argument, as may be shown by examples.† We shall prove, however, that the following is true:‡

THEOREM XIII: *If the absolute value of the remainder after terms of the nth order in the series* (5) *is never greater than $\varphi(n)$, where φ is such a function that*

$$\sum_{n=1}^{\infty}\frac{\varphi(n)\log n}{n}$$

converges and that $\lim_{n=\infty}\varphi(n)\log n = 0$, then the series (6) *converges uniformly.*

These conditions would be satisfied, for example, if $\varphi(n)$ were a constant multiple of $1/(\log^{2+\delta} n)$, δ being any positive quantity.

Let the sum of (5) be denoted by $f(x)$. The Fourier's series

$$\int f(x)\,dx = \sum_{n=1}^{\infty}\frac{1}{n}(a_n \sin nx - b_n \cos nx),$$

* Cf. the proof of Theorem IX*b*.

† Cf., e. g., FEJÉR, *Über gewisse Potenzreihen an der Konvergenzgrenze*, Sitzungsberichte der Bayerischen Akademie (1910), 3. Abhandlung, pp. 10–11.

‡ For related but different theorems, cf. BERNSTEIN, loc. cit., pp. 95–96.

the constant of integration being suitably chosen, converges uniformly; and, as we assure ourselves through the medium of Theorem II, its remainder

$$\rho_n(x) = \sum_{h=n+1}^{\infty} \frac{1}{h}(a_h \sin hx - b_h \cos hx)$$

satisfies the inequality

$$|\rho_n(x)| \leqq \frac{3K\varphi(n)\log n}{n}$$

for $n \geqq 2$. Let

$$R_{np}(x) = \sum_{h=n+1}^{n+p}(a_h \sin hx - b_n \cos hx).$$

Then

$$\begin{aligned} R_{np}(x) &= \sum_{h=n+1}^{n+p} h[\rho_{h-1}(x) - \rho_h(x)] \\ &= \sum_{h=n+1}^{n+p}[(h+1)-h]\rho_h(x) + (n+1)\rho_n(x) - (n+p+1)\rho_{n+p}(x) \\ &= \sum_{h=n+1}^{n+p}\rho_h(x) + (n+1)\rho_n(x) - (n+p+1)\rho_{n+p}(x); \end{aligned}$$

$$|R_{np}(x)| \leqq \sum_{h=n+1}^{\infty}\frac{3K\varphi(h)\log h}{h} + 3K\frac{n+1}{n}\varphi(n)\log n + 3K\frac{n+p+1}{n+p}\varphi(n+p)\log(n+p).$$

Let ϵ be any positive number, and let N be so large that for $n > N$

$$3K\frac{n+1}{n}\varphi(n)\log n < \frac{\epsilon}{3},$$

and

$$\sum_{h=n+1}^{\infty}\frac{3K\varphi(h)\log h}{h} < \frac{\epsilon}{3}.$$

This is surely possible, with the hypotheses that we have adopted. Then we have also

$$3K\frac{n+p+1}{n+p}\varphi(n+p)\log(n+p) < \frac{\epsilon}{3},$$

since $n+p > n$, and this inequality, combined with the preceding ones, shows that

$$|R_{np}(x)| < \epsilon,$$

if $n > N$. It follows that the series (6) does converge uniformly, as was asserted.

With our earlier results at our command, we are in a position to supplement a theorem of my thesis* concerning the existence of derivatives as a necessary

* *Thesis*, Satz XVIII*a*, concerning derivatives of even order.

condition for the possibility of certain degrees of approximation by trigonometric sums, and to prove by our methods a theorem of S. BERNSTEIN* relating to the corresponding problem in polynomial approximation.

We suppose that a function $f(x)$ of period 2π can be represented in the form $\mathfrak{T}_n(x) + O(\varphi(n))$ for all positive integral values of n, and that $n^l \varphi(n) \log^3 n$ is a monotonically decreasing function; this condition is satisfied, for example, if $\varphi(n) = C/(n^l \log^4 n)$, where C is an arbitrary constant.

Since the remainder in the Fourier's series for $f(x)$ is uniformly $O(\varphi(n) \log n)$, it is a consequence of Theorem XIIa, refined in the manner suggested in the remark following it, that $f(x)$ has everywhere a continuous lth derivative when l is even.

If l is odd, the same conclusion can be drawn without alteration in the hypothesis. The integral† $\int f(x)\,dx$ has the form $\mathfrak{T}_n(x) + O(\varphi(n)/n)$, and is represented by its Fourier sum of the nth order with an error which is $O[\varphi(n)(\log n)/n]$. Hence, by the refined Theorem XIIa, the integral has a continuous derivative of order $l+1$, and $f(x)$ has a continuous derivative of order l. To summarize:

THEOREM XIV: *If $f(x)$ can be approximately represented by a trigonometric sum of the nth order at most for all positive integral values of n with an error not exceeding $\varphi(n)$, where φ is such that $n^l \varphi(n) \log^3 n$ decreases monotonically, then $f(x)$ has everywhere a continuous lth derivative.*

It will readily be seen how the hypothesis may be replaced by slightly more general ones, for instance, by the assumption that $n^l \varphi(n) (\log\log n)^2 \log^2 n$ decreases monotonically.

For the polynomial case, suppose that $f(x)$ has the form $\mathfrak{P}_n(x) + O(\varphi(n))$, uniformly throughout some interval $a \leqq x \leqq b$. Without loss of generality, we may assume that the interval is that from -1 to $+1$. We assume again that $n^l \varphi(n) \log^3 n$ decreases monotonically. As $f(x)$ is represented by a polynomial in x, so‡ $f(\cos x)$ is represented by a polynomial in $\cos x$; and a polynomial of the nth degree in $\cos x$ is a trigonometric sum of the nth order in x. Hence $f(\cos x)$ has a continuous lth derivative with respect to x, for all values of x. It follows that $f(\cos x)$ has a continuous derivative with respect to $\cos x$ at all points except possibly those at which $dx/d(\cos x)$ becomes infinite; that is, at all points where $\cos x \neq \pm 1$. Hence we have the theorem:

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*S. BERNSTEIN, Comptes Rendus, vol. 152 (1911), pp. 502–504; cf. *Thesis*, Satz XVIII. The theorem of BERNSTEIN is somewhat more general than that which we shall obtain; the theorem of the thesis is essentially less so, relating only to derivatives of even order.

A detailed discussion, including the trigonometric case, is given by BERNSTEIN in Chapter II, pp. 22–36, of the essay referred to in previous footnotes.

† Here again it may be necessary to begin by subtracting a constant from $f(x)$, an operation which clearly does not impair the correctness of the ultimate result.

‡ Cf. *Thesis*, p. 85.

THEOREM XV: *If $f(x)$ can be approximately represented throughout a closed interval by a polynomial of the nth degree at most for all positive integral values of n with a maximum error not exceeding $\varphi(n)$, where φ is such that $n^l \varphi(n) \log^3 n$ decreases monotonically, then $f(x)$ has a continuous lth derivative throughout the interval, with the possible exception of the end-points.*

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CERTAIN CONTINUOUS DEFORMATIONS OF SURFACES APPLICABLE TO THE QUADRICS*

BY

LUTHER PFAHLER EISENHART

Introduction.

A rectilinear congruence which possesses the property that the asymptotic lines correspond on the focal surfaces is called a W-congruence. Either focal surface of such a congruence admits an infinitesimal deformation such that the direction of deformation at a point is parallel to the normal to the other surface at the corresponding point. BIANCHI† has discovered certain W-congruences whose two focal surfaces are applicable to the same quadric Q; such a congruence may be looked upon as a transformation B_k of one focal surface into the other. We have therefrom infinitesimal deformations of surfaces applicable to quadrics. It is the purpose of this paper to show that it is possible to establish with the aid of these infinitesimal deformations the equations of a continuous deformation of such surfaces and to obtain in intrinsic form the equations of a family of surfaces each of which is a continuous deform of the others and is applicable to a quadric. A family of this sort we call a *system* (Q).‡

The paper is divided into two parts which deal respectively with the cases where Q is a paraboloid or a central quadric. In § 1 are set down certain equations and identities given by BIANCHI and others derived from them. These are applied in § 2 to the expression in analytical form of the infinitesimal deformation of a surface S, applicable to a real hyperbolic paraboloid, which is determined by a transformation B_k of S. These results enable us to find in § 3 that the intrinsic determination of a surface S reduces to the integration of two partial differential equations.

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* Presented to the Society at Cleveland, December 31, 1912. This paper was also presented in part at the Fifth International Congress of Mathematicians in Cambridge, England August 22–28, 1912, and an abstract is printed in the Proceedings of the Congress.

† *Lezioni di Geometria Differenziale*, vol. III, Pisa (1909). Hereafter a reference to this volume will be of the form B, p. ——.

‡ Analogous families of pseudospherical surfaces have been considered by BIANCHI, *Sopra una classe di deformazioni continue delle superficie pseudosferiche*, Annali di Matematica, ser. 3, vol. 18 (1911), pp. 1–67.

In §§ 4, 5 we derive the integrable system of differential equations which determine a system (Q) of non-ruled surfaces S, and in § 6 the similar question is handled for systems (Q) of ruled surfaces.

Since the correspondence between a surface S and a transform established by the associated W-congruence is not also the correspondence of applicability of these surfaces, it is necessary to develop in § 7 certain transformed equations referring to the latter type of correspondence. These are applied in § 8 to give an analytical proof of the reciprocal character of a transformation B_k.

In setting up a system (Q) we have associated with each surface S of the system a surface $\overline{S}$, which arises from S by a transformation B_k. We say that these surfaces form *the system conjugate* to the given one. In § 9 we find that the conjugate system is itself a system (Q) whose conjugate system is the given one. In § 10 the question of generalized transformations B_k from a system (Q) into others of the same sort is investigated.

In §§ 11, 12, 13, 14 we give in condensed form similar equations and results for surfaces S applicable to the real hyperboloid of one sheet. In order to facilitate comparison of analogous equations and identities, we have given them the same numbers in the two parts of the paper.

In § 15 we show by what change of variables and constants it is possible to transform the equations and identities so as to establish the existence of systems (Q) of pseudospherical surfaces, as found by BIANCHI.*

The closing section deals with systems (Q) of ruled surfaces applicable to the hyperboloid of revolution of one sheet and incidentally with a deformation of Bertrand curves into curves of the same kind.

PART I.

SYSTEMS (Q) OF SURFACES APPLICABLE TO A PARABOLOID.

§ 1. *Preliminary Formulas.*

If we take the equations of the paraboloid P in the form

$$(1) \qquad x_0 = \sqrt{p}\,(u+v), \qquad y_0 = \sqrt{q}\,(u-v), \qquad z_0 = 2uv,$$

the generators are parametric and the first fundamental coefficients have the values

$$(2) \qquad E = p+q+4v^2, \qquad F = p-q+4uv, \qquad G = p+q+4u^2.$$

For the sake of brevity we define a function H by the first of the equations

$$(3) \qquad H = \tfrac{1}{4}(EG-F^2) = p\,(u-v)^2 + q\,(u+v)^2 + pq.$$

* Loc. cit.

The functions E, F, G, and H satisfy the following identities which are necessary in the discussion:

$$\frac{\partial E}{\partial v} = 2\frac{\partial F}{\partial u}, \qquad \frac{\partial G}{\partial u} = 2\frac{\partial F}{\partial v},$$

$$E\frac{\partial \log H}{\partial v} + F\frac{\partial \log H}{\partial u} - 2\frac{\partial F}{\partial u} = 0,$$

(4)

$$F\frac{\partial \log H}{\partial v} + G\frac{\partial \log H}{\partial u} - 2\frac{\partial F}{\partial v} = 0,$$

$$\left(\frac{\partial H}{\partial u}\right)^2 + 4pqE = 4(p+q)H,$$

$$\frac{\partial^2 \log H}{\partial u^2} + \frac{1}{2}\left(\frac{\partial \log H}{\partial u}\right)^2 = \frac{2pqE}{H^2},$$

(5)

$$\frac{\partial^2 \log H}{\partial u \partial v} + \frac{1}{2}\frac{\partial \log H}{\partial u}\frac{\partial \log H}{\partial v} = -\frac{2pqF}{H^2},$$

$$\frac{\partial^2 \log H}{\partial v^2} + \frac{1}{2}\left(\frac{\partial \log H}{\partial v}\right)^2 = \frac{2pqG}{H^2}.$$

Moreover, the Christoffel symbols have the following values in this case:

(6) $$\left\{\begin{matrix}11\\1\end{matrix}\right\} = \left\{\begin{matrix}22\\1\end{matrix}\right\} = \left\{\begin{matrix}11\\2\end{matrix}\right\} = \left\{\begin{matrix}22\\2\end{matrix}\right\} = 0, \quad \left\{\begin{matrix}12\\1\end{matrix}\right\} = \frac{1}{2}\frac{\partial \log H}{\partial v}, \quad \left\{\begin{matrix}12\\2\end{matrix}\right\} = \frac{1}{2}\frac{\partial \log H}{\partial u};$$

and consequently the Gauss equations* assume the form

$$\frac{\partial^2 x}{\partial u^2} = DX_3, \qquad \frac{\partial^2 x}{\partial v^2} = D''X_3,$$

(7)

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{1}{2}\frac{\partial \log H}{\partial v}\frac{\partial x}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\frac{\partial x}{\partial v} + D'X_3,$$

where D, D', D'' denote the second fundamental coefficients of a non-ruled surface S, applicable to the paraboloid, and X_3, Y_3, Z_3 are the direction-cosines of the normal to S.

The Gaussian curvature of S is found to have the value

(8) $$K = -\frac{4pq}{H^2} \equiv -\frac{1}{\rho^2},$$

ρ being thus defined. Consequently

(9) $$DD'' - D'^2 = -\frac{4pq}{H}.$$

* E., p. 154. A reference of this sort is to the author's *Differential Geometry*, Ginn and Co., Boston, 1909.

A transformation B_k of S is given analytically by *

$$x_1 = x + l\frac{\partial x}{\partial u} + m\frac{\partial x}{\partial v}, \tag{10}$$

where

$$l = \frac{U}{W}, \qquad m = \frac{V}{W}, \tag{11}$$

and

$$\begin{aligned} U &= 2(\sqrt{qp'} \mp \sqrt{pq'})\lambda^2 u^2 - 2(\sqrt{pq} \mp \sqrt{p'q'})\lambda u \\ &\quad - \frac{k}{2}(\sqrt{qp'} \pm \sqrt{q'p})\lambda^2 + \tfrac{1}{2}(\sqrt{qp'} \mp \sqrt{pq'}), \\ V &= 2(\sqrt{qp'} \pm \sqrt{pq'})\lambda^2 v^2 - 2(\sqrt{qp} \pm \sqrt{p'q'})\lambda v \\ &\quad - \frac{k}{2}(\sqrt{qp'} \mp \sqrt{q'p})\lambda^2 + \tfrac{1}{2}(\sqrt{qp'} \pm \sqrt{pq'}), \end{aligned} \tag{12}$$

$$W = 2\lambda[\sqrt{pq} - \sqrt{p'q}\,(u+v)\lambda \pm \sqrt{pq'}\,(u-v)\lambda],$$

where p', q' and k are constants given by

$$p' = p - k, \qquad q' = q + k, \tag{13}$$

and λ is a function of u and v satisfying the equations

$$\begin{aligned} \frac{\partial\lambda}{\partial u} &= \frac{\sqrt{pq}}{kH}V + \frac{\epsilon}{2k\sqrt{H}}(DU + D'V), \\ \frac{\partial\lambda}{\partial v} &= \frac{\sqrt{pq}}{kH}U + \frac{\epsilon}{2k\sqrt{H}}(D'U + D''V). \end{aligned} \qquad (\epsilon = \pm 1). \tag{14}$$

When the transformation B_k is determined by the generators of the confocal quadric P_k corresponding to those of parameter u on P, the upper sign in (12) must be used and in (14) $\epsilon = +1$. When the other system on P_k is used, we take either the upper signs in (12) and $\epsilon = -1$ in (14), or the lower signs in (12) and $\epsilon = +1$ in (14).†

We define four functions as follows:

$$\begin{aligned} L_0 &= \frac{\partial l}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial v}m + 1, & M_0 &= \frac{\partial m}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}m, \\ P_0 &= \frac{\partial l}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}l, & Q_0 &= \frac{\partial m}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial u}l + 1. \end{aligned} \tag{15}$$

It is important to observe that $\partial l/\partial u, \cdots, \partial m/\partial v$, as used in (15) and

* B., p. 13.

† Cf. B., pp. 13, 88, 319.

hereafter, indicate the derivates of l and m with respect to u or v appearing explicitly in these functions and not as obtained implicitly from λ. Thus we shall always write

$$\frac{d}{du}=\frac{\partial}{\partial u}+\frac{\partial \lambda}{\partial u}\cdot\frac{\partial}{\partial \lambda}, \qquad \frac{d}{dv}=\frac{\partial}{\partial v}+\frac{\partial \lambda}{\partial v}\cdot\frac{\partial}{\partial \lambda}. \tag{16}$$

One shows that

$$L_0 l - P_0 m = 0, \qquad M_0 l - Q_0 m = 0, \tag{17}$$

and consequently (15) may be written

$$\begin{aligned} \frac{L_0}{m}=\frac{P_0}{l}&=\frac{\partial \log l}{\partial v}+\frac{1}{2}\frac{\partial \log H}{\partial v}, \\ \frac{M_0}{m}=\frac{Q_0}{l}&=\frac{\partial \log m}{\partial u}+\frac{1}{2}\frac{\partial \log H}{\partial u}. \end{aligned} \tag{18}$$

These results justify the following definition of a function A,

$$A=\frac{lM_0-mL_0}{m}=\frac{lQ_0-mP_0}{l}. \tag{19}$$

Substituting from (15) we find the following expressions for A:

$$\begin{aligned} A &= \frac{l}{m}\frac{\partial m}{\partial u}-\frac{\partial l}{\partial u}+\frac{1}{2}\frac{\partial \log H}{\partial u}l-\frac{1}{2}\frac{\partial \log H}{\partial v}m-1, \\ &= \frac{\partial m}{\partial v}-\frac{m}{l}\frac{\partial l}{\partial v}+\frac{1}{2}\frac{\partial \log H}{\partial u}l-\frac{1}{2}\frac{\partial \log H}{\partial v}m+1, \\ &= \frac{l}{m}\frac{\partial m}{\partial u}-\frac{m}{l}\frac{\partial l}{\partial v}+\frac{l}{2}\frac{\partial \log H}{\partial u}-\frac{m}{2}\frac{\partial \log H}{\partial v}. \end{aligned} \tag{20}$$

As an immediate result we have the identities

$$\frac{\partial l}{\partial u}-\frac{m}{l}\frac{\partial l}{\partial v}+1=0, \qquad \frac{l}{m}\frac{\partial m}{\partial u}-\frac{\partial m}{\partial v}-1=0. \tag{21}$$

Bianchi* has established the following important identity:

$$m\frac{\partial l}{\partial \lambda}-l\frac{\partial m}{\partial \lambda}=\frac{H}{\sqrt{pq}}\frac{k}{W}A, \tag{22}$$

or, in other form,

$$V\frac{\partial U}{\partial \lambda}-U\frac{\partial V}{\partial \lambda}=\frac{H}{\sqrt{pq}}kWA. \tag{23}$$

* B., p. 19.

In consequence of (22) we may define a function J thus

$$J = \left(\frac{kH}{\sqrt{pq}W}\frac{L_0}{m} + \frac{\partial l}{\partial \lambda}\right)m = \left(\frac{kH}{\sqrt{pq}W}\frac{M_0}{m} + \frac{\partial m}{\partial \lambda}\right)l. \tag{24}$$

If $X_1, Y_1, Z_1; X_2, Y_2, Z_2$ denote the direction-cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$ respectively on S, we have

$$\frac{\partial x}{\partial u} = \sqrt{E}X_1, \qquad \frac{\partial x}{\partial v} = \sqrt{G}X_2, \tag{25}$$

and similar expressions for $Y_1, \cdots Z_2$.

In consequence of (7) and (4) we have

$$\begin{aligned} &\frac{\partial X_1}{\partial u} = \frac{D}{\sqrt{E}}X_3, \quad \frac{\partial X_1}{\partial v} = \frac{1}{2E}\frac{\partial \log H}{\partial u}(-FX_1 + \sqrt{EG}X_2) + \frac{D'}{\sqrt{E}}X_3, \\ &\frac{\partial X_2}{\partial u} = \frac{1}{2G}\frac{\partial \log H}{\partial v}(\sqrt{EG}X_1 - FX_2) + \frac{D'}{\sqrt{G}}X_3, \quad \frac{\partial X_2}{\partial v} = \frac{D''}{\sqrt{G}}X_3, \end{aligned} \tag{26}$$

and the derivatives of X_3 are given by*

$$\begin{aligned} \frac{\partial X_3}{\partial u} &= \frac{FD' - GD}{4H}\sqrt{E}X_1 + \frac{FD - ED'}{4H}\sqrt{G}X_2, \\ \frac{\partial X_3}{\partial v} &= \frac{FD'' - GD'}{4H}\sqrt{E}X_1 + \frac{FD' - ED''}{4H}\sqrt{G}X_2. \end{aligned} \tag{27}$$

Moreover, the Codazzi equations for S assume the form

$$\begin{aligned} \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} &= \frac{1}{2}\frac{\partial \log H}{\partial v}D + \frac{1}{2}\frac{\partial \log H}{\partial u}D', \\ \frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} &= \frac{1}{2}\frac{\partial \log H}{\partial v}D' + \frac{1}{2}\frac{\partial \log H}{\partial u}D''. \end{aligned} \tag{28}$$

§ 2. *Infinitesimal Deformation of S.*

The equations of an infinitesimal deformation of S are†

$$x' = x + \epsilon\xi, \qquad y' = y + \epsilon\eta, \qquad z' = z + \epsilon\zeta, \tag{29}$$

where ϵ is an infinitesimal constant and ξ, η, ζ are functions of u and v satisfying the conditions

$$\Sigma\frac{\partial x}{\partial u}\frac{\partial \xi}{\partial u} = 0, \qquad \Sigma\frac{\partial x}{\partial v}\frac{\partial \xi}{\partial v} = 0, \qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial \xi}{\partial v} + \Sigma\frac{\partial x}{\partial v}\frac{\partial \xi}{\partial u} = 0. \tag{30}$$

* E., p. 154.

† E., pp. 373, 374.

We know that if S and S_1 are the focal sheets of any W-congruence,* an infinitesimal deformation of S is given by taking ξ, η, ζ proportional to the direction-cosines of the normal to S_1 at the corresponding point, the factor of proportionality being determined by the conditions (30). A fundamental property of the transformation B_k is that S and its transform S_1, given by (10), are the focal surfaces of the W-congruence formed by the joins of corresponding points. Hence the knowledge of a transformation B_k of S leads to an infinitesimal deformation of S. We shall investigate this deformation.

The direction-cosines of the normal to S_1 are proportional to expressions of the form †

$$(31) \qquad -(Fl+Gm)\sqrt{E}X_1+(El+Fm)\sqrt{G}X_2+\frac{2\epsilon H^{\frac{1}{2}}AX_3}{\sqrt{pq}}.$$

Accordingly we put

$$(32) \quad \xi = e^T\left[-(Fl+Gm)\sqrt{E}X_1+(El+Fm)\sqrt{G}X_2+\frac{2\epsilon H^{\frac{1}{2}}A}{\sqrt{pq}}X_3\right],$$

and determine T subject to the above conditions. From (32) we obtain with the aid of (26) and (27)

$$\begin{aligned}\frac{\partial\xi}{\partial u} &= e^T\sqrt{E}X_1\left[-(Fl+Gm)\left(\frac{\partial T}{\partial u}+\frac{1}{2}\frac{\partial \log H}{\partial u}\right)-\frac{\partial F}{\partial v}m-\left(F\frac{dl}{du}+G\frac{dm}{du}\right)\right.\\ &\quad\left.+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD'-GD)\right]+e^T\sqrt{G}X_2\left[(El+Fm)\left(\frac{\partial T}{\partial u}+\frac{1}{2}\frac{\partial \log H}{\partial u}\right)\right.\\ &\quad\left.+\frac{\partial F}{\partial u}m+\left(E\frac{dl}{du}+F\frac{dm}{du}\right)+\epsilon\frac{\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD-ED')\right]\\ &\quad+X_3\left[\frac{2\epsilon}{\sqrt{pq}}\frac{d}{du}(e^TH^{\frac{1}{2}}A)-(Fl+Gm)e^TD+(El+Fm)e^TD'\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial\xi}{\partial v} &= e^T\sqrt{E}X_1\left[-(Fl+Gm)\left(\frac{\partial T}{\partial v}+\frac{1}{2}\frac{\partial \log H}{\partial v}\right)-\frac{\partial F}{\partial v}l\right.\\ &\quad\left.-\left(F\frac{dl}{dv}+G\frac{dm}{dv}\right)+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD''-GD')\right]\\ &\quad+e^T\sqrt{G}X_2\left[(El+Fm)\left(\frac{\partial T}{\partial v}+\frac{1}{2}\frac{\partial \log H}{\partial v}\right)+\frac{\partial F}{\partial u}l+\left(E\frac{dl}{dv}+F\frac{dm}{dv}\right)\right.\\ &\quad\left.+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD'-ED'')\right]\\ &\quad+X_3\left[\frac{2\epsilon}{\sqrt{pq}}\frac{d}{dv}(e^TH^{\frac{1}{2}}A)-(Fl+Gm)e^TD'+(El+Fm)e^TD''\right].\end{aligned} \qquad (33)$$

* E., p. 420.

† B., p. 44.

When these and similar values for $\partial\eta/\partial u, \cdots, \partial\zeta/\partial v$ are substituted in (30), we obtain the following equations for the determination of T:

$$
\begin{aligned}
&\frac{\partial T}{\partial u}+\frac{\partial \log H}{\partial u}+\frac{d \log m}{du}+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}\frac{D}{m}=0,\\
&\frac{\partial T}{\partial v}+\frac{\partial \log H}{\partial v}+\frac{d \log l}{dv}-\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}\frac{D''}{l}=0,\\
&l\frac{\partial T}{\partial u}-m\frac{\partial T}{\partial v}+\frac{dl}{du}-\frac{dm}{dv}-\frac{\epsilon\sqrt{H}}{\sqrt{pq}}AD'=0.
\end{aligned}
\tag{34}
$$

By means of (14) the first two of equations (34) are reducible to

$$
\begin{aligned}
&\frac{\partial T}{\partial u}+\frac{\partial \log H}{\partial u}+\frac{\partial \log m}{\partial u}+\frac{\sqrt{pq}W}{kH}\frac{\partial m}{\partial\lambda}+\frac{\epsilon W}{2k\sqrt{H}}\left(D\frac{\partial l}{\partial\lambda}+D'\frac{\partial m}{\partial\lambda}\right)=0,\\
&\frac{\partial T}{\partial v}+\frac{\partial \log H}{\partial v}+\frac{\partial \log l}{\partial v}+\frac{\sqrt{pq}W}{kH}\frac{\partial l}{\partial\lambda}+\frac{\epsilon W}{2k\sqrt{H}}\left(D'\frac{\partial l}{\partial\lambda}+D''\frac{\partial m}{\partial\lambda}\right)=0,
\end{aligned}
\tag{35}
$$

and the third of (34) is found to be a consequence of the other two.

With the aid of the Codazzi equations (28) it can be shown that equations (35) satisfy the condition of integrability and consequently we have the theorem:

When a transformation B_k of a surface S applicable to the paraboloid P is known, an infinitesimal deformation of S is given by a quadrature.

§ 3. *Intrinsic Determination of Surfaces Applicable to the Paraboloid.*

With the aid of the foregoing results we are able to give an intrinsic determination of surfaces applicable to the paraboloid. To this end we observe that if the values of D, D', D'' given by (34) be substituted in equations (14) we obtain the single equation

$$
\begin{aligned}
&l\left(\frac{\partial T}{\partial u}+\frac{\partial \log H}{\partial u}+\frac{\partial \log m}{\partial u}+\frac{\partial \log l}{\partial\lambda}\frac{\partial\lambda}{\partial u}\right)\\
&\qquad+m\left(\frac{\partial T}{\partial v}+\frac{\partial \log H}{\partial v}+\frac{\partial \log l}{\partial v}+\frac{\partial \log m}{\partial\lambda}\frac{\partial\lambda}{\partial v}\right)=0.
\end{aligned}
\tag{36}
$$

In consequence of this equation the last of equations (34) may be given each of the forms

$$
\begin{aligned}
&\frac{\partial T}{\partial v}+\frac{A}{m}+\frac{\partial \log H}{\partial v}+\frac{1}{l}\frac{\partial l}{\partial v}+\frac{1}{m}\frac{\partial m}{\partial\lambda}\frac{\partial\lambda}{\partial v}+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{AD'}{2m}=0,\\
&\frac{\partial T}{\partial u}-\frac{A}{l}+\frac{\partial \log H}{\partial u}+\frac{1}{m}\frac{\partial m}{\partial u}+\frac{1}{l}\frac{\partial l}{\partial\lambda}\frac{\partial\lambda}{\partial u}-\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{AD'}{2l}=0.
\end{aligned}
\tag{37}
$$

By means of these expressions for D' and those for D and D'' given by (34), the Gauss equation (9) may be reduced to

$$\left(\frac{\partial\lambda}{\partial u}\frac{\partial T}{\partial v}-\frac{\partial\lambda}{\partial v}\frac{\partial T}{\partial u}\right)+\left(\frac{\partial \log Hl}{\partial v}+\frac{W\sqrt{pq}}{Hk}\frac{\partial l}{\partial\lambda}\right)\frac{\partial\lambda}{\partial u}$$

$$-\left(\frac{\partial}{\partial u}\log Hm+\frac{W\sqrt{pq}}{Hk}\frac{\partial m}{\partial\lambda}\right)\frac{\partial\lambda}{\partial v}+\frac{\sqrt{pq}U}{Hk}\frac{\partial T}{\partial u}-\frac{\sqrt{pq}V}{Hk}\frac{\partial T}{\partial v} \tag{38}$$

$$+\left[\frac{U\sqrt{pq}}{Hk}\left(\frac{\partial \log Hm}{\partial u}\right)-\frac{V\sqrt{pq}}{Hk}\left(\frac{\partial \log Hl}{\partial v}\right)\right]=0.$$

Again, when the expression (34) for D and the first of (37) for D' are substituted in the first of the Codazzi equations (28), the resulting equation is of the same general form as (38). If the function $\frac{\partial\lambda}{\partial u}\frac{\partial T}{\partial v}-\frac{\partial\lambda}{\partial v}\frac{\partial T}{\partial u}$ be eliminated from it by means of (38), we obtain

$$\left(\frac{\partial \log H}{\partial v}+\frac{\partial}{\partial v}\log\frac{A}{m}+\frac{\sqrt{pq}U}{Hk}\frac{\partial}{\partial\lambda}\log\frac{A}{m}\right)\left(\frac{\partial T}{\partial u}+\frac{\partial \log Hm}{\partial u}+\frac{\partial \log l}{\partial\lambda}\right)$$

$$-\left(\frac{\partial \log H}{\partial u}+\frac{\sqrt{pq}V}{Hk}\frac{\partial}{\partial\lambda}\log\frac{A}{m}\right)\left(\frac{\partial T}{\partial v}+\frac{\partial \log Hl}{\partial v}+\frac{\partial \log m}{\partial\lambda}\right)=0.$$

On substituting the first expression for A, as given by (20), this reduces to

$$\left(\frac{\partial}{\partial v}\log\frac{l}{m}+\frac{1}{2}\frac{\partial \log H}{\partial v}+\frac{A}{m}-\frac{\sqrt{pq}W}{Hk}\frac{\partial l}{\partial\lambda}\right)\left(\frac{\partial T}{\partial u}+\frac{\partial \log Hm}{\partial u}+\frac{\partial \log l}{\partial\lambda}\right)$$

$$+\left(\frac{1}{2}\frac{\partial \log H}{\partial v}-\frac{\sqrt{pq}W}{Hk}\frac{\partial m}{\partial\lambda}\right)\left(\frac{\partial T}{\partial v}+\frac{\partial}{\partial v}\log Hl+\frac{\partial \log m}{\partial\lambda}\right)=0.$$

This in turn is reducible by means of (20) and (22) to (36).

In like manner it may be shown that the second of the Codazzi equations is satisfied when λ and T are solutions of (36) and (38). Hence we have the following theorem:

Given two equations (36), (38) *in which the functions* H, l, m, U, V, W *have the forms defined in* § 1; *if* λ *and* T *constitute any integral of these equations, the functions* D, D', D'' *given by* (34) *and* (37), *and the functions* E, F, G *given by* (2) *define intrinsically a surface applicable to the paraboloid. Furthermore, when a surface* S *has been found in this way and the coördinates* x, y, z *are known, a second surface of the same kind is given directly by* (10).

Later (§ 7) we shall see in what way the intrinsic equations of the second surface can be found without quadrature.

§ 4. *Continuous Deformation of S.*

We are not interested primarily in the infinitesimal deformations of S, but rather in showing that with the aid of the equations of such deformations it is possible to discover systems of surfaces arising from continuous deformations of S.

To this end we replace ξ, η, ζ in (32) by $\partial x/\partial w, \partial y/\partial w, \partial z/\partial w$, where w is a third variable, and we assume that the functions D, D', D'', λ involve this variable as well as u and v. Now we replace (32) by

$$\frac{\partial x}{\partial w} = e^{\tau}\left[-(Fl+Gm)\sqrt{E}X_1+(El+Fm)\sqrt{G}X_2+\frac{2\epsilon H^{3/2}}{\sqrt{pq}}AX_3\right], \tag{39}$$

and because of (14) and (25) equations (33) may be replaced by

$$\begin{aligned}\frac{\partial X_1}{\partial w} &= e^{\tau}(FX_1-\sqrt{EG}X_2)+\frac{B}{\sqrt{E}}X_3,\\ \frac{\partial X_2}{\partial w} &= e^{\tau}(\sqrt{EG}X_1-FX_2)+\frac{C}{\sqrt{G}}X_3,\end{aligned} \tag{40}$$

where we have put for the sake of brevity

$$\begin{aligned}B &= \frac{2\epsilon}{\sqrt{pq}}\frac{d}{du}(e^{\tau}H^{3/2}A)-e^{\tau}[D(Fl+Gm)-D'(El+Fm)],\\ C &= \frac{2\epsilon}{\sqrt{pq}}\frac{d}{dv}(e^{\tau}H^{3/2}A)-e^{\tau}[D'(Fl+Gm)-D''(El+Fm)].\end{aligned} \tag{41}$$

By means of the identities

$$\Sigma X_1\frac{\partial X_3}{\partial w}+\Sigma X_3\frac{\partial X_1}{\partial w}=0, \qquad \Sigma X_2\frac{\partial X_3}{\partial w}+\Sigma X_3\frac{\partial X_2}{\partial w}=0,$$

we find that

$$\frac{\partial X_3}{\partial w} = \frac{\sqrt{EG}}{4H}\left[(FX_1-\sqrt{EG}X_2)\frac{C}{\sqrt{G}}-(\sqrt{EG}X_1-FX_2)\frac{B}{\sqrt{E}}\right]. \tag{42}$$

In order that these equations be consistent, it is necessary and sufficient that the following conditions be satisfied for the X's, Y's and Z's:

$$\frac{\partial}{\partial u}\left(\frac{\partial X_i}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{\partial X_i}{\partial u}\right) \qquad (i=1,2,3), \tag{43}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial X_i}{\partial w}\right)=\frac{\partial}{\partial w}\left(\frac{\partial X_i}{\partial u}\right), \qquad \frac{\partial}{\partial v}\left(\frac{\partial X_i}{\partial w}\right)=\frac{\partial}{\partial w}\left(\frac{\partial X_i}{\partial v}\right) \quad (i=1,2,3). \tag{44}$$

Since equations (43) are true for each surface S, it is necessary to consider only equations (44). The first of these for $i=1$ leads to an equation of the form

$aX_1 + bX_2 + cX_3 = 0$ and to two other equations obtained by replacing the X's by Y's and Z's respectively. These equations are equivalent to $a = b = c = 0$. By substitution these are found to reduce to the following

$$\begin{aligned} &\frac{D'B - DC}{4H} + e^T\left(\frac{\partial T}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right) = 0, \\ &\frac{\partial D}{\partial w} = e^T(FD - ED') + \frac{dB}{du}. \end{aligned} \tag{45}$$

In like manner the first of (44) for $i = 2$ leads to the first of (45) and to

$$\frac{\partial D'}{\partial w} = e^T(GD - FD') + \frac{dC}{du} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u}. \tag{46}$$

Furthermore, the second of equations (44) for $i = 1, 2$ give rise to

$$\begin{aligned} &\frac{BD'' - CD'}{4H} + e^T\left(\frac{\partial T}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}\right) = 0, \\ &\frac{\partial D'}{\partial w} = e^T(FD' - ED'') + \frac{dB}{dv} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u}, \\ &\frac{\partial D''}{\partial w} = e^T(GD' - FD'') + \frac{dC}{dv}. \end{aligned} \tag{47}$$

Finally, no new equations are introduced by equations (44) for $i = 3$,

In consequence of (9) the first of equations (45) and (47) are equivalent to

$$\begin{aligned} B &= e^T\frac{H^2}{pq}\left[D\left(\frac{\partial T}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}\right) - D'\left(\frac{\partial T}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right)\right], \\ C &= e^T\frac{H^2}{pq}\left[D'\left(\frac{\partial T}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}\right) - D''\left(\frac{\partial T}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right)\right]. \end{aligned} \tag{48}$$

By means of (35) these may be given the form

$$\begin{aligned} B &= \frac{\sqrt{HW}}{k}e^T\left[2\epsilon\frac{\partial m}{\partial \lambda} + \frac{\sqrt{HJ}}{\sqrt{pq}}\left(\frac{D'}{l} - \frac{D}{m}\right)\right], \\ C &= \frac{\sqrt{HW}}{k}e^T\left[-2\epsilon\frac{\partial l}{\partial \lambda} + \frac{\sqrt{HJ}}{\sqrt{pq}}\left(\frac{D''}{l} - \frac{D'}{m}\right)\right]. \end{aligned} \tag{49}$$

When these expressions are compared with (41), we find that in order that they be consistent it is necessary and sufficient that

$$El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq} = \frac{2HWJ}{k\sqrt{pq}}. \tag{50}$$

Later it will be shown that this is an identity.

Again, in order that the two expressions for $\partial D'/\partial w$ in (46) and (47) be equivalent, it is necessary that

$$\frac{dC}{du} - \frac{dB}{dv} + e^T(GD + ED'' - 2FD') = 0. \tag{51}$$

But when the values of B and C from (41) are substituted in this equation it is satisfied identically, by virtue of the Codazzi equations. Hence the only conditions introduced by equations (44) are

$$\begin{aligned}
\frac{\partial D}{\partial w} &= e^T(FD - ED') + \frac{dB}{du}, \\
\frac{\partial D'}{\partial w} &= e^T(GD - FD') + \frac{dC}{du} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u} \\
&= e^T(FD' - ED'') + \frac{dB}{dv} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u}, \\
\frac{\partial D''}{\partial w} &= e^T(GD' - FD'') + \frac{dC}{dv}.
\end{aligned} \tag{52}$$

Reviewing the case rapidly, we observe that in addition to (52) the equations of condition of the problem are (9), (14), (28) and (34). It must now be shown that these equations in λ, T, D, D', D'' are consistent. To this end we observe that if equations (9) and (28) be differentiated with respect to w and in the result the derivatives of D, D', D'' be replaced by the expressions (52), in which B and C are given the values (48), the resulting equations can be shown to be satisfied identically in consequence of (9) and (28) and of equations resulting from their differentiation with respect to u or v.

§ 5. *Intrinsic Determination of Systems* (Q).

Before considering the determination of functions D, D', D'', λ, T satisfying the above mentioned conditions, we call attention to the fact that these equations constitute the necessary and sufficient condition that the following system be completely integrable:

$$\begin{aligned}
\frac{\partial x}{\partial u} &= \sqrt{E}X_1, \qquad \frac{\partial x}{\partial v} = \sqrt{G}X_2, \\
\frac{\partial x}{\partial w} &= e^T\left[-(Fl + Gm)\sqrt{E}X_1 + (El + Fm)\sqrt{G}X_2 + \frac{2\epsilon H^2}{\sqrt{pq}}AX_3\right], \\
\frac{\partial X_1}{\partial u} &= \frac{D}{\sqrt{E}}X_3, \qquad \frac{\partial X_1}{\partial v} = \frac{1}{2E}\frac{\partial \log H}{\partial u}(-FX_1 + \sqrt{EG}X_2) + \frac{D'}{\sqrt{E}}X_3,
\end{aligned}$$

$$\frac{\partial X_2}{\partial u} = \frac{1}{2G}\frac{\partial \log H}{\partial v}(\sqrt{EG}X_1 - FX_2) + \frac{D'}{\sqrt{G}}X_3, \qquad \frac{\partial X_2}{\partial v} = \frac{D''}{\sqrt{G}}X_3,$$

$$\frac{\partial X_3}{\partial u} = \frac{FD' - GD}{4H}\sqrt{E}X_1 + \frac{FD - ED'}{4H}\sqrt{G}X_2, \tag{53}$$

$$\frac{\partial X_3}{\partial v} = \frac{FD'' - GD'}{4H}\sqrt{E}X_1 + \frac{FD' - ED''}{4H}\sqrt{G}X_2,$$

$$\frac{\partial X_1}{\partial w} = e^{\tau}(FX_1 - \sqrt{EG}X_2) + \frac{B}{\sqrt{E}}X_3, \qquad \frac{\partial X_2}{\partial w} = e^{\tau}(\sqrt{EG}X_1 - FX_2) + \frac{C}{\sqrt{G}}X_3,$$

$$\frac{\partial X_3}{\partial w} = \frac{\sqrt{EG}}{4H}\left[(FX_1 - \sqrt{EG}X_2)\frac{C}{\sqrt{G}} - (\sqrt{EG}X_1 - FX_2)\frac{B}{\sqrt{E}}\right].$$

Hence from the general theory of triple systems of surfaces in space we know that, when these conditions are satisfied, the quadratic form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 - 8Hme^{\tau}dudw + 8Hle^{\tau}dudw + 4e^{2\tau}H\left(El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq}\right)dw^2, \tag{54}$$

(which arises from the first three of (53) and similar expressions in y and z) defines space referred to a triple system of surfaces, such that the surfaces $w = \text{const.}$ are applicable to one another and to the paraboloid (1).

We return now to the consideration of the system of consistent equations (9), (14), (28), (34) and (49). It was shown in § 3 that by means of (35) and (37) the functions D, D', D'' may be eliminated from (9), (14) and (28) with the result that we obtain the two equations (36) and (38). When the expressions for D, D', D'' from (35) and (37) are substituted in (49), we obtain

$$B = \frac{2\epsilon\sqrt{H}We^{\tau}}{k}\left[\frac{\partial m}{\partial \lambda} + \frac{2J}{A}\left(\frac{\partial T}{\partial u} + \frac{\partial \log H}{\partial u} + \frac{\partial \log m}{\partial u} - \frac{A}{2l} + \frac{1}{2}\frac{\partial \log lm}{\partial \lambda}\frac{\partial \lambda}{\partial u}\right)\right],$$

$$C = -\frac{2\epsilon\sqrt{H}We^{\tau}}{k}\left[\frac{\partial l}{\partial \lambda} - \frac{2J}{A}\left(\frac{\partial T}{\partial v} + \frac{\partial \log H}{\partial v} + \frac{\partial \log l}{\partial v} + \frac{A}{2m} + \frac{1}{2}\frac{\partial \log lm}{\partial \lambda}\frac{\partial \lambda}{\partial v}\right)\right]. \tag{55}$$

With the aid of these expressions and the former ones equations (52) are transformable into four equations of the form

$$
\begin{aligned}
\frac{\partial^2 T}{\partial u\,\partial w} &= A_{11}\frac{\partial^2 T}{\partial u^2} + B_{11}\frac{\partial^2 \lambda}{\partial u^2} + A_{12}\frac{\partial^2 T}{\partial u\,\partial v} + B_{12}\frac{\partial^2 \lambda}{\partial u\,\partial v} + E_1,\\
\frac{\partial^2 \lambda}{\partial u\,\partial w} &= C_{11}\frac{\partial^2 T}{\partial u^2} + D_{11}\frac{\partial^2 \lambda}{\partial u^2} + C_{12}\frac{\partial^2 T}{\partial u\,\partial v} + D_{12}\frac{\partial^2 \lambda}{\partial u\,\partial v} + F_1,\\
\frac{\partial^2 T}{\partial v\,\partial w} &= A_{21}\frac{\partial^2 T}{\partial u\,\partial v} + B_{21}\frac{\partial^2 \lambda}{\partial u\,\partial v} + A_{22}\frac{\partial^2 T}{\partial v^2} + B_{22}\frac{\partial^2 \lambda}{\partial v^2} + E_2,\\
\frac{\partial^2 \lambda}{\partial v\,\partial w} &= C_{21}\frac{\partial^2 T}{\partial u\,\partial v} + D_{21}\frac{\partial^2 \lambda}{\partial u\,\partial v} + C_{22}\frac{\partial^2 T}{\partial v^2} + D_{22}\frac{\partial^2 \lambda}{\partial v^2} + F_2,
\end{aligned}
\tag{56}
$$

where $A_{11}, \cdots, F_1, F_2$ are determinate functions of u, v, λ, T and the first derivatives of λ and T with respect to u, v and w. It is unnecessary to write down the explicit expressions for these functions, but it is important to remark that since the system of equations (9), (14), (28), (34) and (49) is consistent so also is the system (36), (38) and (56). Furthermore, when a set of values of the latter system is known, one obtains the functions D, D', D'' directly from (35) and (37).

The systems of equations is such that the formal integration reduces to the determination of power series in w, thus

$$
\begin{aligned}
\lambda &= \varphi_0 + \varphi_1 w + \varphi_2 w^2 + \cdots\\
T &= \psi_0 + \psi_1 w + \psi_2 w^2 + \cdots
\end{aligned}
\tag{57}
$$

where the φ's and ψ's are functions of u and v. Evidently for φ_0 and ψ_0 we take a set of solutions of (36) and (38). To find φ_1 and ψ_1 we substitute the expressions (57) in (56), and put $w = 0$; this gives four equations of the form

$$
\frac{\partial \varphi_1}{\partial u} = A_1, \qquad \frac{\partial \varphi_1}{\partial v} = A_2, \qquad \frac{\partial \psi_1}{\partial u} = B_1, \qquad \frac{\partial \psi_1}{\partial v} = B_2,
\tag{58}
$$

where the functions A and B involve $\varphi_1, \psi_1, \varphi_0, \psi_0$ and the derivatives of φ_0 and ψ_0. When one has a set of functions φ_1, ψ_1, satisfying these equations, the determination of φ_2 and ψ_2 is a similar problem. In this case we differentiate equations (56) with respect to w, substitute (57) and put $w = 0$. Proceeding step by step we reduce the problem of finding the coefficients in (57) to the integration of systems of equations of the type (58). It is not our purpose to go further into the consideration of the character and domain of the solutions, but we have given sufficient indication to enable us to state that equations (36), (38), (56) possess common solutions. Hence we have the theorem:

There exist triple systems of surfaces such that the surfaces in one family are continuous deforms of one another and of the hyperbolic paraboloid.

We have thus established the existence of *systems* (Q).

Thus far we have considered only surfaces applicable to the hyperbolic paraboloid. Bianchi has derived the equations for transformations of surfaces applicable to the *elliptic paraboloid** and also to the *imaginary paraboloid*†

$$\frac{x^2}{p}+\frac{y^2}{q}=2iz. \tag{59}$$

The changes to be made in the formulas in these two cases are such that one sees readily that by a repetition of the processes of the foregoing sections one can easily establish the existence of systems (Q) of surfaces applicable to these two types of paraboloids also. As a matter of fact we have shown elsewhere‡ that there exist systems (Q) of surfaces applicable to the paraboloid (59), the process of proof being less direct than the foregoing.

§ 6. *Systems* (Q) *of Ruled Surfaces.*

Thus far we have excluded the case where S is ruled. We consider it now, and observe that (9) may be replaced by

$$D=0, \qquad D'=-2\frac{\sqrt{pq}}{\sqrt{H}}, \qquad D''=2\sqrt{H}\varphi, \tag{60}$$

where φ is independent of u, the assumption being that the lines $v=$ const. are straight. From geometrical considerations it is evident that if the transform S_1 given by (10) is to be ruled also, the upper signs in (12) must be used and $\epsilon=+1$ in (14). The latter equations reduce in this case to

$$\frac{\partial\lambda}{\partial u}=0, \qquad \frac{\partial\lambda}{\partial v}=\frac{V\varphi(v)}{k}, \tag{61}$$

consequently λ is a function of v alone. Instead of (35) we have the consistent set

$$\begin{gathered}\frac{\partial T}{\partial u}+\frac{\partial\log H}{\partial u}+\frac{\partial\log m}{\partial u}=0,\\ \frac{\partial T}{\partial v}+\frac{\partial\log H}{\partial v}+\frac{\partial\log l}{\partial v}+\frac{\varphi(v)\,W}{k}\frac{\partial m}{\partial\lambda}=0,\end{gathered} \tag{62}$$

and the functions B and C which appear in (40) have the form

* B., p. 113.

† B., p. 150.

‡ *Sopra le deformazioni continue delle superficie reali applicabili sul paraboloide a parametro puramente immaginario*, Rendiconti della R. Accademia dei Lincei, vol. XXI (1912), pp. 458–462.

$$B = -\frac{2H^{3/2}e^T}{\sqrt{pq}}\left(\frac{\partial \log m}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right),$$

$$C = \frac{2H^{3/2}e^T}{\sqrt{pq}}\left(\frac{\partial \log l}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v} - \frac{2\sqrt{pq}m\varphi}{k}\right). \tag{63}$$

When these expressions and the values from (60) are substituted in (52), the first two of the latter become

$$\frac{\partial D}{\partial w} = 0, \qquad \frac{\partial D'}{\partial w} = 0, \tag{64}$$

and the third one is reducible to

$$\sqrt{pq}\frac{\partial \varphi}{\partial w} + \frac{2\sqrt{pq}mHe^T}{k}\frac{\partial \varphi}{\partial v} + \varphi He^T\left[\frac{\sqrt{pq}}{H}\left(F + \frac{mG}{l}\right) - \frac{H}{\sqrt{pq}}\frac{L_0}{m}\frac{A}{l} + \frac{2\sqrt{pq}}{kW}\frac{\partial V}{\partial v}\right] = 0, \tag{65}$$

in consequence of (5) and of the following identities which are readily established:

$$\frac{\partial^2 \log m}{\partial u^2} = \left(\frac{\partial \log m}{\partial u}\right)^2, \qquad \frac{\partial^2 \log l}{\partial v^2} = \left(\frac{\partial \log l}{\partial v}\right)^2,$$

$$\frac{\partial^2 \log m}{\partial u\,\partial v} = \frac{\partial^2 \log l}{\partial u\,\partial v} = \frac{\partial \log m}{\partial u}\cdot\frac{\partial \log l}{\partial v}. \tag{66}$$

Since φ is at most a function of v and w, the coefficients in equation (65) must be independent of u. The first is constant and the second is evidently independent of u, in consequence of (61) and (62). As regards the coefficient of φ we observe that it may be written

$$e^T Hm\left[\frac{\sqrt{pq}}{H}\left(\frac{F}{m} + \frac{G}{l}\right) - \frac{H}{\sqrt{pq}}\frac{L_0}{m^2}\frac{A}{l} + \frac{2\sqrt{pq}}{k}\frac{1}{V}\frac{\partial V}{\partial v}\right],$$

and consequently we must show that

$$\frac{\sqrt{pq}}{H}\left(\frac{F}{m} + \frac{G}{l}\right) - \frac{H}{\sqrt{pq}}\frac{L_0}{m^2}\frac{A}{l}$$

is independent of u. In view of (19) and (50) this may be written

$$-\frac{\sqrt{pq}}{Hm}\left[E\frac{l}{m} + F + \frac{4H}{k} + \frac{H^2}{pq}\cdot\frac{A}{m}\cdot\frac{M_0}{m}\right].$$

If we differentiate this expression with respect to u and make use of (4) and (5), the result is reducible to

$$-\frac{H}{\sqrt{pq}}\left[H\left(\frac{\partial \log m}{\partial u}\right)^2+\frac{\partial H}{\partial u}\frac{\partial \log m}{\partial u}+p+q+\frac{\partial m}{\partial u}\cdot\frac{4}{k}pq\right],$$

and by direct substitution it is found that this expression vanishes identically.

Consider now equations (62). The first may be replaced by

$$T=\log\frac{f}{Hm}, \tag{67}$$

where f is independent of u. Substituting this value in the second, we obtain

$$\frac{\partial f}{\partial v}=\frac{V'}{V}f, \tag{68}$$

where V' denotes the derivative of V with respect to v entering explicitly and not implicitly by means of λ. Moreover, equation (65) may be written

$$\frac{\partial \varphi}{\partial w}+\frac{2}{k}f\frac{\partial \varphi}{\partial v}+f\sigma\varphi=0, \tag{69}$$

where σ is a determinate function of u, v and λ.

Equations (61), (68), (69) form a system to which the standard existence theorems apply. Hence recalling the general discussion of § 4, we have the result:

There exist systems (Q) *for which all of the surfaces* $w=$ *const. are ruled, and the determination of the intrinsic equations of such systems requires the integration of a system of partial differential equations of the first order.*

§ 7. *Change of Parameters. Fundamental Identities.*

We will now establish the fundamental identity (50), and to this end consider the transformation B_k defined by equations of the type

$$\bar{x}=x+l\frac{\partial x}{\partial u}+m\frac{\partial x}{\partial v}. \tag{70}$$

From this we have by differentiation

$$\begin{aligned}\frac{\partial \bar{x}}{\partial u}&=L_0\frac{\partial x}{\partial u}+M_0\frac{\partial x}{\partial v}+\left(\frac{\partial l}{\partial \lambda}\frac{\partial x}{\partial u}+\frac{\partial m}{\partial \lambda}\frac{\partial x}{\partial v}\right)\frac{\partial \lambda}{\partial u}+(Dl+D'm)X_3,\\ \frac{\partial \bar{x}}{\partial v}&=P_0\frac{\partial x}{\partial u}+Q_0\frac{\partial x}{\partial v}+\left(\frac{\partial l}{\partial \lambda}\frac{\partial x}{\partial u}+\frac{\partial m}{\partial \lambda}\frac{\partial x}{\partial v}\right)\frac{\partial \lambda}{\partial v}+(D'l+D''m)X_3,\end{aligned} \tag{71}$$

where L_0, M_0, P_0, Q_0 are defined by (15).

By means of (71) we can find the first fundamental coefficients of $\bar{S}$, but

as Bianchi* has remarked, these functions have not the same form as (2). He proved,† however, that it is possible to transform the parameters on $\bar{S}$ into a new system $\bar{u}$, $\bar{v}$ so that the new coefficients $\bar{E}$, $\bar{F}$, $\bar{G}$ as functions of $\bar{u}$ and $\bar{v}$ have the form (2). In fact, this change of parameters is made in accordance with the affine transformation of Ivory, whose analytic form is

$$\bar{u} = \frac{\sqrt{qp'}\,(u+v) + \sqrt{pq'}\,(u-v) - \sqrt{pq}\,(4uv+k)\lambda}{2\,[\sqrt{pq'}\,(u-v)\lambda - \sqrt{qp'}\,(u+v)\lambda + \sqrt{pq}\,]},$$
$$\bar{v} = \frac{1}{2\lambda}. \tag{72}$$

By differentiation we have

$$\frac{d\bar{u}}{du} = 4\sqrt{pq}\lambda^2\frac{V}{W^2} + \frac{\partial u_1}{\partial\lambda}\frac{\partial\lambda}{\partial u},$$
$$\frac{d\bar{u}}{dv} = 4\sqrt{pq}\lambda^2\frac{U}{W^2} + \frac{\partial u_1}{\partial\lambda}\frac{\partial\lambda}{\partial v}, \tag{73}$$

and

$$\frac{d\bar{v}}{du} = -\frac{1}{2\lambda^2}\frac{\partial\lambda}{\partial u}, \qquad \frac{d\bar{v}}{dv} = -\frac{1}{2\lambda^2}\frac{\partial\lambda}{\partial v}. \tag{74}$$

By means of (14) we obtain

$$\delta = \frac{d(\bar{u},\bar{v})}{d(u,v)} = \frac{\epsilon\sqrt{pq}}{Hk}(Dl^2 - D''m^2). \tag{75}$$

From these results and the expression‡

$$\frac{\partial\bar{u}}{\partial\lambda} = -\frac{2k\lambda^2}{W^2}H, \tag{76}$$

we get from (71)

$$\frac{\partial\bar{x}}{\partial\bar{u}} = \frac{1}{4\lambda^2}\frac{W}{\sqrt{pq}}\left[\frac{L_0}{m}\frac{\partial x}{\partial u} + \frac{M_0}{m}\frac{\partial x}{\partial v} - \frac{\epsilon\sqrt{pq}}{\sqrt{H}}2X_3\right],$$
$$\frac{\partial\bar{x}}{\partial\bar{v}} = -2\lambda^2\left[\left(\frac{J}{m} - \frac{1}{2}\frac{kH}{W\sqrt{pq}}\frac{L_0}{m}\right)\frac{\partial x}{\partial u} + \left(\frac{J}{l} - \frac{1}{2}\frac{kH}{W\sqrt{pq}}\frac{M_0}{m}\right)\frac{\partial x}{\partial v} + \epsilon k\frac{\sqrt{H}}{W}X_3\right]. \tag{77}$$

If for the sake of brevity we put

$$K_1 = L_0^2E + 2L_0M_0F + M_0^2G + \frac{4pqm^2}{H},$$
$$K_2 = \frac{L_0E + M_0F}{m} + \frac{FL_0 + GM_0}{l}, \tag{78}$$

* B., pp. 25, 76–79.
† B., p. 32.
‡ B., p. 35.

we have from (77) and (24)

$$\bar{E} = \frac{1}{16\lambda^4}\frac{W^2}{pqm^2}K_1, \qquad \bar{F} = -\left(\frac{J}{m}K_2 - \frac{1}{2}\frac{kH}{W\sqrt{pq}m^2}K_1\right)\frac{W}{2\sqrt{pq}},$$

(79)

$$\bar{G} = 4\lambda^4\left[(El^2 + 2Flm + Gm^2)\frac{J^2}{l^2m^2} - K_2\frac{kH}{W\sqrt{pq}}\frac{J}{m} + \frac{1}{4}\frac{k^2H^2}{W^2pqm^2}K_1\right].$$

With the aid of these values we find

$$4\bar{H} = \bar{E}\bar{G} - \bar{F}^2 = \frac{W^2}{H^2m^2}\left(El^2 + 2Flm + Gm^2 + \frac{A^2H^2}{pq}\right)J^2. \tag{80}$$

Bianchi has established* the identity

$$El^2 + 2Flm + Gm^2 + \frac{1}{k^2W^2}\left(V\frac{\partial U}{\partial\lambda} - U\frac{\partial V}{\partial\lambda}\right)^2 = \frac{H\bar{H}}{pq}, \tag{81}$$

which in consequence of (23) may be written

$$El^2 + 2Flm + Gm^2 + \frac{A^2H^2}{pq} = \frac{H\bar{H}}{pq}. \tag{82}$$

Comparing this with (80) we see that we must have

$$J = \epsilon'\frac{2lm\sqrt{pq}}{W},$$

where $\epsilon' = \pm 1$, to be determined. From this result and (82) it follows that, if (50) is an identity, so also is

$$\bar{H} = 4\epsilon'\frac{lmpq}{k} \tag{83}$$

and conversely. Making use of (13) and the form of $\bar{H}$ analogous to (3), we find that the preceding equation is identically satisfied when ϵ' is -1. Hence we have

$$J = -\frac{2lm\sqrt{pq}}{W}, \tag{84}$$

and the identity (50) is established. The latter may now be written, in consequence of (84),

$$El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq} = -\frac{4Hlm}{k}. \tag{85}$$

To these may be added the following useful identities which arise from (24) and (84)

* B., p. 46.

$$(86)\qquad \frac{\partial l}{\partial \lambda} = -\frac{\sqrt{pq}}{W}\left(\frac{kH}{pq}\frac{L_0}{m} + 2l\right), \qquad \frac{\partial m}{\partial \lambda} = -\frac{\sqrt{pq}}{W}\left(\frac{kH}{pq}\frac{M_0}{m} + 2m\right).*$$

The foregoing results enable us to find the expressions for the second fundamental coefficients of $\bar{S}$ in simple form. From (31) and (32) it follows that the direction-cosines of the normal to $\bar{S}$ are of the form

$$(87)\qquad \bar{X}_3 = Re^{-\tau}\xi, \quad \text{where} \quad R^{-2} = El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq}.$$

From these we have by means of (40)

$$(88)\qquad \begin{aligned} \frac{\partial \bar{X}_3}{\partial u} &= R\left[\sqrt{E}\left(FX_1 - \sqrt{EG}X_2\right) + Be^{-\tau}X_3\right] + \bar{X}_3\frac{\partial}{\partial u}\log Re^{-\tau}, \\ \frac{\partial \bar{X}_3}{\partial v} &= R\left[\sqrt{G}\left(\sqrt{EG}X_1 - FX_2\right) + Ce^{-\tau}X_3\right] + \bar{X}_3\frac{\partial}{\partial v}\log Re^{-\tau}. \end{aligned}$$

In consequence of (14) and (86) equations (71) may be given the form

$$(89)\qquad \begin{aligned} \frac{\partial \bar{x}}{\partial u} &= \left[-\frac{2pqlm}{kH} + \frac{\epsilon}{2k\sqrt{H}}(DU + D'V)\frac{\partial l}{\partial \lambda}\right]\sqrt{E}X_1 \\ &\quad + \left[-\frac{2pqm^2}{kH} + \frac{\epsilon}{2k\sqrt{H}}(DU + D'V)\frac{\partial m}{\partial \lambda}\right]\sqrt{G}X_2 + (Dl + D'm)X_3, \\ \frac{\partial \bar{x}}{\partial v} &= \left[-\frac{2pql^2}{kH} + \frac{\epsilon}{2kH}(D'U + D''V)\frac{\partial l}{\partial \lambda}\right]\sqrt{E}X_1 \\ &\quad + \left[-\frac{2pqlm}{kH} + \frac{\epsilon}{2k\sqrt{H}}(D'U + D''V)\frac{\partial m}{\partial \lambda}\right]\sqrt{G}X_2 + (D'l + D''m)X_3. \end{aligned}$$

From these expressions we find

$$(90)\qquad \bar{D} = -\Sigma\frac{\partial \bar{X}_3}{\partial u}\frac{\partial \bar{x}}{\partial u} = \sigma D, \qquad \bar{D}' = \sigma D', \qquad \bar{D}'' = \sigma D'',$$

where

$$(91)\qquad \sigma = -\frac{2RH}{k}(Dl^2 - D''m^2).$$

Hence the asymptotic lines on S and $\bar{S}$ correspond and we have the following result, established by Bianchi in another manner:†

The surfaces S and $\bar{S}$, the latter resulting from a transformation B_k of the former, are the focal surfaces of a W-congruence formed by the lines joining corresponding points.

* If the other system of generators of the quadric Q_k are used, so that we take the lower signs in the expressions for U and V, it merely comes to changing the sign of every radical under which q' appears. So far as the present result goes, it is the same in both cases, as one sees readily by reviewing the above work.

† B., p. 89.

We have not written down the first fundamental coefficients E, F, G referring to the parameters u and v on $\bar{S}$, but one gets them at once from (71). Hence, when one has a surface S defined intrinsically as explained in § 3, a transform $\bar{S}$ is given intrinsically at once.

§ 8. *The Inverse of a Transformation B_k.*

Since S and $\bar{S}$ are the focal sheets of a congruence, we have

$$x = \bar{x} + \bar{l}\frac{\partial \bar{x}}{\partial \bar{u}} + \bar{m}\frac{\partial \bar{x}}{\partial \bar{v}}. \tag{92}$$

Substituting this expression in (70) and making use of (77), we obtain an equation of the form

$$a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} + cX_3 = 0, \tag{93}$$

where a, b, c are determinate expressions which must vanish identically because y and z also satisfy (93). These identities are

$$l + \frac{\bar{l}}{4\lambda^2}\frac{W}{\sqrt{pq}}\frac{L_0}{m} + \bar{m}\lambda^2\left(\frac{kH}{\sqrt{pq}W}\frac{L_0}{m} + \frac{4\sqrt{pq}l}{W}\right) = 0,$$

$$m + \frac{\bar{l}}{4\lambda^2}\frac{W}{\sqrt{pq}}\frac{M_0}{m} + \bar{m}\lambda^2\left(\frac{kH}{\sqrt{pq}W}\frac{M_0}{m} + \frac{4\sqrt{pq}m}{W}\right) = 0,$$

$$\frac{\bar{l}W}{4\lambda^2} + \frac{\bar{m}kH\lambda^2}{W} = 0,$$

which are equivalent to

$$\bar{l} = \frac{kH\lambda^2}{\sqrt{pq}W}, \qquad \bar{m} = -\frac{W}{4\sqrt{pq}\lambda^2}. \tag{94}$$

From these follows

$$H = -\frac{4pq\bar{l}\bar{m}}{k}, \tag{95}$$

which is analogous to (83) where $\epsilon' = -1$.

We wish to show that in fact S is a transform B_k of $\bar{S}$. To this end we put for brevity

$$A' = \sqrt{qp'} + \sqrt{pq'}, \qquad B' = \sqrt{qp'} - \sqrt{pq'},$$

with the result that (72) becomes

$$\bar{u} = \frac{A'u + B'v - \sqrt{pq}\lambda(4uv + k)}{(\sqrt{pq} - B'u\lambda - A'v\lambda)^2}.$$

If we put $v = 1/2\bar{\lambda}$ and solve the foregoing equation with respect to u, we obtain

$$u = \frac{A'\bar{u} + B'\bar{v} - \sqrt{pq}\bar{\lambda}\,(4\bar{u}\bar{v} + k)}{(\sqrt{pq} - B'\bar{u}\bar{\lambda} - A\bar{v}\bar{\lambda})^2},$$

which shows the reciprocal character of the transformation (72).

If we take

$$\bar{l} = \frac{\bar{U}}{\bar{W}}, \qquad \bar{m} = \frac{\bar{V}}{\bar{W}},$$

where $\bar{U}$, $\bar{V}$, $\bar{W}$ are the functions obtained on replacing u, v, λ in (12) by $\bar{u}$, $\bar{v}$, $\bar{\lambda}$, by means of the foregoing results these expressions are reducible to the form (94). Hence we have established in an analytical manner the following theorem which Bianchi* discovered by geometrical considerations:

If $\bar{S}$ is obtained from S by a transformation B_k, S may be obtained from $\bar{S}$ by another transformation $\bar{B}_k$, the inverse of the former.

§ 9. *The Conjugate System* (Q).

Suppose that we have a system (Q) and that upon each surface S of the system we effect a transformation B_k, given by

$$x_1 = x + l_1\frac{\partial x}{\partial u} + m_1\frac{\partial x}{\partial v}, \tag{96}$$

where l_1 and m_1 are given by replacing λ and k in (11) and (12) by λ_1 and k_1.

If (96) be differentiated with respect to w, the result is reducible by (39) and (40) to

$$\begin{aligned}\frac{dx_1}{dw} = {} & X_1\sqrt{E}\left\{e^{\tau}[F(l_1 - l) + G(m_1 - m)] + \frac{\partial l_1}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial w}\right\} \\ & + X_2\sqrt{G}\left\{-e^{\tau}[E(l_1 - l) + F(m_1 - m)] + \frac{\partial m_1}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial w}\right\} \\ & + X_3\left\{e^{\tau}\frac{2\epsilon H^3 A}{\sqrt{pq}} + l_1 B + m_1 C\right\}.\end{aligned} \tag{97}$$

It should be observed that in the functions B and C there appear l and m, but not l_1 and m_1.

From the given system (Q) we obtain by means of (96) a triple system of surfaces expressed in terms of parameters u, v, w, such that the surfaces $w = \text{const.}$ are applicable to the paraboloid. If we wish this system to be given in terms of the parameters u_1, v_1, w, where u_1 and v_1 are obtained from

* B., p. 36.

(72) by replacing λ and k by λ_1 and k_1, the direction of the curves for which both u_1 and v_1 are constant is given by

$$\frac{dx_1}{dw} - \frac{\partial x_1}{\partial u_1}\frac{\partial u_1}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial w} - \frac{\partial x_1}{\partial v_1}\frac{\partial v_1}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial w} \equiv \frac{\partial x_1}{\partial w}.$$

From (97), (77) and (76) we find

$$\begin{aligned}\frac{\partial x_1}{\partial w} &= X_1\sqrt{E}e^r[F(l_1-l)+G(m_1-m)]\\ &\quad - X_2\sqrt{G}e^r[E(l_1-l)+F(m_1-m)]\\ &\quad + X_3\left[e^r\frac{2\epsilon H^{\frac{1}{2}}A}{\sqrt{pq}} + l_1B + m_1C - \frac{2\epsilon_1 k_1\sqrt{H}}{W_1}\frac{\partial \lambda_1}{\partial w}\right],\end{aligned} \tag{98}$$

ϵ_1 being ± 1 and not necessarily equal to ϵ (it arises from (77)).

Before considering a general system of surfaces S_1 thus defined, we will look at the surfaces $\bar{S}$ which are used in constructing the system (Q). For this particular case we have

$$l_1 = l, \quad m_1 = m, \quad \lambda_1 = \lambda, \quad k_1 = k, \quad \epsilon_1 = \epsilon, \quad u_1 = \bar{u}, \quad v_1 = \bar{v},$$

and so (98) reduces to

$$\frac{\partial \bar{x}}{\partial w} = 2\left[\frac{e^r H}{k}(Dl^2 - D''m^2) - \frac{\epsilon k\sqrt{H}}{W}\frac{\partial \lambda}{\partial w}\right]X_3. \tag{99}$$

Hence a tangent to a curve of parameter w is parallel to the normal to S at the corresponding point.

The parameters $\bar{u}$ and $\bar{v}$ were chosen so that the surfaces $\bar{S}$ are seen to be applicable to one another and to the paraboloid. The relation between a pair of surfaces S and $\bar{S}$ is reciprocal, as shown in § 8. The curves along which the surfaces $\bar{S}$ are deformed into one another are such that a tangent to such a curve is parallel to the corresponding normal to S. Hence if we say that the surfaces $\bar{S}$ form a system *conjugate* to the given system (Q), we have the result

The system conjugate to a given system (Q) is itself a system (Q).

In order to give further consequences of the foregoing results, we observe that in consequence of (81) we may replace (39) by

$$\frac{\partial x}{\partial w} = e^r R^{-1}\bar{X}_3, \qquad R^{-2} = \frac{4H^2\bar{H}}{pq} \tag{100}$$

If in like manner we put in accordance with (99)

$$\frac{\partial \bar{x}}{\partial w} = e^r \bar{R}^{-1} X_3,$$

we have from (77) and with the aid of (94)

$$\Sigma \frac{\partial \bar{x}}{\partial \bar{u}} \frac{\partial \bar{x}}{\partial w} = 4\epsilon \bar{H} \bar{m} e^{r}, \qquad \Sigma \frac{\partial \bar{x}}{\partial \bar{v}} \frac{\partial \bar{x}}{\partial w} = -4\epsilon \bar{H} \bar{l} e^{r},$$

provided that

$$\frac{1}{\bar{R}^2} = \frac{4H\bar{H}^2}{pq}, \tag{101}$$

which evidently is the analogue use of (100), and from the preceding results we know that it is true.

§ 10. *Transformations of Systems* (Q).

We return to the consideration of equations (96) and (98). The former when applied to a system (Q) leads to a triple system of surfaces such that each of the surfaces $w = \text{const.}$ is applicable to the paraboloid, by a suitable change of parameters involving λ_1. It is natural to inquire whether it is possible to determine λ_1 involving w in such a way that the transformed surfaces shall form a system (Q).

If such a transformation is possible, the function λ_1 must satisfy the equations

$$\begin{aligned} \frac{\partial \lambda_1}{\partial u} &= \frac{\sqrt{pq}}{k_1 H} V_1 + \frac{\epsilon_1}{2k_1 H}(DU_1 + D' V_1), \\ \frac{\partial \lambda_1}{\partial v} &= \frac{\sqrt{pq}}{k_1 H} U_1 + \frac{\epsilon_1}{2k_1 H}(D' U_1 + D'' V_1), \\ \frac{\partial \lambda_1}{\partial w} &= \frac{W_1 \epsilon_1}{2k_1 \sqrt{H}}(l_1 B + m_1 C) + \frac{e^r W_1}{H k_1}\pi, \end{aligned} \tag{102}$$

where π is a function to be determined.

Referring to (54) we observe that in consequence of (85) the condition

$$\Sigma \frac{\partial x}{\partial u} \frac{\partial x}{\partial w} \cdot \Sigma \frac{\partial x}{\partial v} \frac{\partial x}{\partial w} = k\Sigma \left(\frac{\partial x}{\partial w}\right)^2 \tag{103}$$

must be satisfied by a system (Q). From the results obtained in certain special cases it is probable that this condition is sufficient. Applying it to the surfaces S_1 we obtain a quadratic equation in π, namely

$$(k_1 - k)\pi^2 + P\pi + Q = 0, \tag{104}$$

where P and Q are determinate functions of $u, v, \lambda, \lambda_1, k$ and k_1.

The conditions of integrability of equations (102) lead to four equations of the form

$$\frac{\partial \pi}{\partial u} + \xi_1 \pi + \eta_1 = 0, \qquad \frac{\partial \pi}{\partial v} + \xi_2 \pi + \eta_2 = 0,$$
$$\frac{\partial \pi}{\partial \lambda} + \xi_3 \pi + \eta_3 = 0, \qquad \frac{\partial \pi}{\partial \lambda_1} + \xi_4 \pi + \eta_4 = 0, \tag{105}$$

where the functions ξ_i, η_i are determinate. Hence if a function π exists which satisfies (104) and (105), it may be found by differentiation.

We have shown elsewhere* by other methods that these transformed systems (Q) exist for the case where the surfaces S are applicable to the imaginary paraboloid. Since there is no essential difference in the formulas of transformations B_k when the fundamental quadric is a hyperbolic or an imaginary paraboloid, we have good reason to believe that a function π exists satisfying (104) and (105), and that the function λ_1 given by (102) leads to a transformed system (Q). Incidentally we remark that, in the case where the surfaces S are applicable to the imaginary paraboloid, for the curves of deformation of the surfaces S_1 the tangents are parallel to the normals to surfaces $\bar{S}_1$ applicable to the same paraboloid, and each surface $\bar{S}_1$ forms with the corresponding surfaces $S, \bar{S}, S_1$ a quartern in accordance with the "theorem of permutability" which Bianchi† has established for transformations B_k.

PART II.

SYSTEMS (Q) OF SURFACES APPLICABLE TO A CENTRAL QUADRIC.

§ 11. *Preliminary Formulas.*

In the following sections we consider systems (Q) of surfaces S applicable to the hyperboloid

$$x_0 = a\frac{1+uv}{u+v}, \qquad y_0 = b\frac{u-v}{u+v}, \qquad z_0 = c\frac{1-uv}{u+v}, \tag{1}$$

the parameters referring to the generators. From these we obtain

$$E = [(a^2+c^2)v^4 + 2(c^2-a^2+2b^2)v^2 + a^2+c^2](u+v)^{-4},$$
$$F = [(a^2+c^2)u^2v^2 + (c^2-a^2)(u^2+v^2) - 4b^2uv + a^2+c^2](u+v)^{-4}, \tag{2}$$
$$G = [(a^2+c^2)u^4 + 2(c^2-a^2+2b^2)u^2 + a^2+c^2](u+v)^{-4},$$

and

$$H = \tfrac{1}{4}(EG-F^2) = \frac{a^2b^2(1-uv)^2 + b^2c^2(1+uv)^2 + a^2c^2(u-v)^2}{(u+v)^6}. \tag{3}$$

* *Sopra le deformazioni*, etc., l. c., p. 461.

† B., p. 156–179.

It is not our purpose to repeat the details for systems (Q) now under discussion, but merely to give the fundamental equations and identities and state the results. In general there are equations and expressions analogous to all of those in Part I, and in giving certain of them we will use the same numbers as in the former part, so that the reader may compare them and fill in the gaps.

We find readily that

$$K = -\frac{a^2 b^2 c^2}{(u+v)^8 H^2} \equiv -\frac{1}{\rho^2}, \tag{8}$$

and

$$DD'' - D'^2 = -\frac{4a^2 b^2 c^2}{H(u+v)^8}. \tag{9}$$

A transformation B_k of S is given analytically by

$$x_1 = x + l\frac{\partial x}{\partial u} + m\frac{\partial x}{\partial v}, \tag{10}$$

with*

$$l = (u+v)\frac{U}{W}, \qquad m = (u+v)\frac{V}{W}, \tag{11}$$

the various functions being defined by

$$\begin{aligned} U &= \left(\pm\frac{ac}{a'c'} - \frac{b}{b'}\right)2u\cos\theta + \left(\pm\frac{bc}{b'c'} - \frac{a}{a'}\right)(u^2-1)\sin\theta \\ &\quad + \left(\frac{ab}{a'b'} \mp \frac{c}{c'}\right)(u^2+1), \\ V &= \left(\mp\frac{ac}{a'c'} - \frac{b}{b'}\right)2v\cos\theta + \left(\pm\frac{bc}{b'c'} + \frac{a}{a'}\right)(v^2-1)\sin\theta \\ &\quad + \left(\frac{ab}{a'b'} \pm \frac{c}{c'}\right)(v^2+1), \\ W &= 2\left[\pm\frac{ac}{a'c'}(u-v)\cos\theta \mp \frac{bc}{b'c'}(1+uv)\sin\theta + \frac{ab}{a'b'}(1-uv)\right], \end{aligned} \tag{12}$$

where the constants are in the relations

$$a' = \sqrt{a^2+k}, \qquad b' = \sqrt{b^2+k}, \qquad c' = \sqrt{c^2-k}, \tag{13}$$

and θ is a function of u and v such that†

$$\begin{aligned} \frac{\partial\theta}{\partial u} &= \frac{a'b'c'}{k(u+v)^2\rho}V + \epsilon\frac{a'b'c'}{2k\sqrt{abc}\sqrt{\rho}}(DU + D'V), \\ \frac{\partial\theta}{\partial v} &= \frac{a'b'c'}{k(u+v)^2\rho}U + \epsilon\frac{a'b'c'}{2k\sqrt{abc}\sqrt{\rho}}(D'U + D''V). \end{aligned} \tag{14}$$

* B., p. 49.

† B., p. 103.

Here the signs in (12) and ϵ in (14) have a significance analogous to that in Part I.

For the sake of brevity we write

$$L_0 = \frac{\partial l}{\partial u} - \frac{2}{u+v} l + \frac{1}{2}\frac{\partial \log \rho}{\partial v} m + 1, \qquad M_0 = \frac{\partial m}{\partial u} + \frac{1}{2}\frac{\partial \log \rho}{\partial u} m,$$

(15)

$$P_0 = \frac{\partial l}{\partial v} + \frac{1}{2}\frac{\partial \log \rho}{\partial v} l, \qquad Q_0 = \frac{\partial m}{\partial v} - \frac{2}{u+v} m + \frac{1}{2}\frac{\partial \log \rho}{\partial u} l + 1,$$

it being understood that the following notation is used:

$$\frac{d}{du} = \frac{\partial}{\partial u} + \frac{\partial \theta}{\partial u} \cdot \frac{\partial}{\partial \theta}, \qquad \frac{d}{dv} = \frac{\partial}{\partial v} + \frac{\partial \theta}{\partial v} \cdot \frac{\partial}{\partial \theta}. \tag{16}$$

We make use also of the function A defined by

$$A = \frac{lM_0 - mL_0}{m} = \frac{lQ_0 - mP_0}{l}, \tag{19}$$

and we may show that

$$A = \pm \frac{a'b'c'}{k(u+v)^2} \cdot \frac{V}{m\rho}\left(m\frac{\partial l}{\partial \theta} - l\frac{\partial m}{\partial \theta}\right). \tag{22}$$

We are thus enabled to define a function J by

$$J = m\left(\frac{\partial l}{\partial \theta} + \frac{k\rho(u+v)^2}{a'b'c'}\frac{L_0}{V}\right) = l\left(\frac{\partial m}{\partial \theta_0} + \frac{k\rho(u+v)^2}{a'b'c'}\frac{M_0}{V}\right). \tag{24}$$

For the present case we have equations of the same form as (25), (27) § 1 and if in (26) § 1 we replace H by ρ we obtain equations valid for surfaces applicable to the hyperboloid.

§ 12. *Infinitesimal Deformations of S. Intrinsic Determination of Surfaces Applicable to the Hyperboloids.*

The equations

$$x' = x + e\xi. \qquad y' = y + e\eta, \qquad z' = z + e\zeta \tag{29}$$

define an infinitesimal deformation of S, where e denotes an infinitesimal constant, and ξ, η, ζ are functions of the form*

$$\xi = e^T\left[-(Fl+Gm)\sqrt{E}X_1 + (El+Fm)\sqrt{G}X_2 + \frac{2\epsilon\sqrt{abc}\rho^{\frac{3}{2}}}{(u+v)^2}AX_3\right], \tag{32}$$

provided that T satisfies the equations

* In this formula $\epsilon = \pm 1$.

$$\frac{\partial T}{\partial u}-\frac{2}{u+v}+\frac{\partial \log \rho}{\partial u}+\frac{d}{du}\log m+\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{2(u+v)^2H}\frac{D}{m}=0,$$

$$\frac{\partial T}{\partial v}-\frac{2}{u+v}+\frac{\partial \log \rho}{\partial v}+\frac{d}{dv}\log l-\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{2(u+v)^2H}\frac{D''}{l}=0, \tag{34}$$

$$l\left(\frac{\partial T}{\partial u}-\frac{4}{u+v}\right)-m\left(\frac{\partial T}{\partial v}-\frac{4}{u+v}\right)+\frac{dl}{du}-\frac{dm}{dv}-\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{(u+v)^2H}D'=0.$$

By means of (14) the first two of these equations are reducible to

$$\frac{\partial T}{\partial u}-\frac{2}{u+v}+\frac{\partial}{\partial u}\log \rho m + \frac{a'b'c'W}{\kappa\rho(u+v)^3}\left[\frac{\partial m}{\partial \theta}+\frac{\epsilon(u+v)^2\sqrt{\rho}}{2\sqrt{abc}}\left(D\frac{\partial l}{\partial \theta}+D'\frac{\partial m}{\partial \theta}\right)\right]=0,$$

$$\frac{\partial T}{\partial v}-\frac{2}{u+v}+\frac{\partial}{\partial v}\log \rho l + \frac{a'b'c'W}{\kappa\rho(u+v)^3}\left[\frac{\partial l}{\partial \theta}+\frac{\epsilon(u+v)^2\sqrt{\rho}}{2\sqrt{abc}}\left(D'\frac{\partial l}{\partial \theta}+D''\frac{\partial m}{\partial \theta}\right)\right]=0, \tag{35}$$

and the third is a consequence of these.

As the conditions of integrability of (35) are satisfied, we have the theorem:

When a transformation B_k of a surface S applicable to a hyperboloid is known, an infinitesimal deformation of S is given by a quadrature.

As in the case of surfaces applicable to the paraboloids, we can give an intrinsic definition of surfaces applicable to a hyperboloid as follows:

Consider the two differential equations

$$l\left(\frac{\partial T}{\partial u}+\frac{\partial \log \rho}{\partial u}+\frac{\partial \log m}{\partial u}-\frac{2}{u+v}\right) + m\left(\frac{\partial T}{\partial v}+\frac{\partial \log \rho}{\partial v}+\frac{\partial \log l}{\partial v}-\frac{2}{u+v}\right)+\frac{\partial l}{\partial \theta}\frac{\partial \theta}{\partial u}+\frac{\partial m}{\partial \theta}\frac{\partial \theta}{\partial v}=0,$$

$$l\left(\frac{\partial T}{\partial u}+\frac{\partial \log \rho}{\partial u}+\frac{\partial \log m}{\partial u}-\frac{2}{u+v}\right)\left(\frac{k(u+v)^2}{a'b'c'U}\frac{\partial \theta}{\partial v}-1\right) - m\left(\frac{\partial T}{\partial v}+\frac{\partial \log \rho}{\partial v}+\frac{\partial \log l}{\partial v}-\frac{2}{u+v}\right)\left(\frac{k(u+v)^2}{a'b'c'V}\frac{\partial \theta}{\partial u}-1\right) + \left(\frac{\partial m}{\partial \theta}\frac{\partial \theta}{\partial v}-\frac{\partial l}{\partial \theta}\frac{\partial \theta}{\partial u}\right)=0, \tag{38}$$

in which ρ, l, m, U, V, W are given by (8), (11), (12), If T and θ are any solution of these equations, the functions D and D'', given by (34), and D',

given by either of the equivalent expressions

$$\frac{\partial T}{\partial u}-\frac{A}{l}+\frac{\partial \log \rho}{\partial u}-\frac{2}{u+v}+\frac{\partial \log m}{\partial u}+\frac{1}{l}\frac{\partial l}{\partial \theta}\frac{\partial \theta}{\partial u}-\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{2(u+v)^2 Hl}D'=0,$$

(37)

$$\frac{\partial T}{\partial v}+\frac{A}{m}+\frac{\partial \log \rho}{\partial v}-\frac{2}{u+v}+\frac{\partial \log l}{\partial v}+\frac{1}{m}\frac{\partial m}{\partial \theta}\frac{\partial \theta}{\partial v}+\frac{\epsilon\sqrt{abc}\rho^{\frac{3}{2}}A}{2(u+v)^2 Hm}D'=0,$$

are the second fundamental coefficients of a surface applicable to the hyperboloid (1), the first fundamental coefficients being given by (2). Furthermore, when one has a surface S defined in this manner, one can obtain without quadratures the intrinsic equations of a second surface of the same kind, which is a transform under a B_k of S.

§ 13. *Continuous Deformation of S. Intrinsic Determination of Systems (Q) of Surfaces Applicable to a Hyperboloid. Systems (Q) of Ruled Surfaces.*

In view of the results of the preceding section we inquire under what conditions the equations

$$\frac{\partial x}{\partial u}=\sqrt{E}X_1, \qquad \frac{\partial X}{\partial v}=\sqrt{G}X_2,$$

(39)

$$\frac{\partial x}{\partial w}=e^T\left[-(Fl+Gm)\sqrt{E}X_1+(El+Fm)\sqrt{G}X_2+\frac{2\epsilon\sqrt{abc}\rho^{\frac{3}{2}}A}{(u+v)^2}X_3\right]$$

and similar ones in y and z define space referred to a triple system of surfaces, such that the surfaces $w=$ const. are applicable to one another and to the hyperboloid (1).

The discussion of this problem may be carried on as in § 4, with equations similar to (40), (42), (43) and (44), with the difference that now

$$B=2\epsilon\sqrt{abc}\frac{d}{du}\left(\frac{\rho^{\frac{3}{2}}}{(u+v)^2}Ae^T\right)+[-(Fl+Gm)D+(El+Fm)D']e^T,$$

(41)

$$C=2\epsilon\sqrt{abc}\frac{d}{dv}\left(\frac{\rho^{\frac{3}{2}}}{(u+v)^2}Ae^T\right)+[-(Fl+Gm)D'+(El+Fm)D'']e^T.$$

Proceeding as in § 4, we find that the expressions (41) must be equivalent to

$$B=\frac{a'b'c'}{k(u+v)^5}We^T\sqrt{\rho}\left[2\epsilon\sqrt{abc}\frac{\partial m}{\partial \theta}+\sqrt{\rho}(u+v)^2 J\left(\frac{D'}{l}-\frac{D}{m}\right)\right],$$

(49)

$$C=\frac{a'b'c'}{k(u+v)^5}We^T\sqrt{\rho}\left[-2\epsilon\sqrt{abc}\frac{\partial l}{\partial \theta}+\sqrt{\rho}(u+v)^2 J\left(\frac{D''}{l}-\frac{D'}{m}\right)\right],$$

which necessitates that the condition

$$El^2+2Flm+Gm^2+\rho^2A^2=\frac{2a'b'c'W\rho}{k(u+v)^3}J \tag{50}$$

be satisfied; that this is an identity is shown in § 14.

Also the following equations must be consistent with one another and must be satisfied:

$$
\begin{aligned}
\frac{\partial D}{\partial w} &= (DF - D'E)\,e^T + \frac{dB}{du} + \frac{2B}{u+v}, \\
\frac{\partial D'}{\partial w} &= (DG - D'F)\,e^T + \frac{dC}{du} - \frac{B}{2}\frac{\partial \log \rho}{\partial v} - \frac{C}{2}\frac{\partial \log \rho}{\partial u} \\
&= (D'F - D''E)\,e^T + \frac{dB}{dv} - \frac{B}{2}\frac{\partial \log \rho}{\partial v} - \frac{C}{2}\frac{\partial \log \rho}{\partial u}, \\
\frac{\partial D''}{\partial w} &= (D'G - D''F)\,e^T + \frac{dC}{dv} + \frac{2C}{u+v}.
\end{aligned} \tag{52}
$$

As in § 4 it can be shown readily that all the equations of condition of the problem, viz., (9), (14), (34), (52) and the Codazzi equations for S, are consistent with one another. Furthermore, when there exists a set of functions D, D', D'', θ, T satisfying these equations, equations similar to (53) § 5 are satisfied and hence there exists a triple system of surfaces of the kind sought, with respect to which the linear element of space has the form

$$
\begin{aligned}
ds^2 = {} & Edu^2 + 2Fdudv + Gdv^2 + 8He^T(ldv - mdu)\,dw \\
& + 4He^{2T}(El^2 + 2Flm + Gm^2 + \rho^2 A^2)\,dw^2.
\end{aligned} \tag{54}
$$

As regards the existence and determination of systems (Q) of this sort the analytical procedure is similar to that followed in § 5, and the result is the same, namely

There exist triple systems of surfaces such that the surfaces in one family are applicable to one another and to the hyperboloid.

With the aid of the results of Bianchi* one can readily extend the foregoing investigation and establish the existence of systems (Q) of surfaces applicable to any central quadric.

Thus far we have tacitly assumed that the surfaces S are not ruled, but by considerations similar to those of § 6 it may be shown that

There exist systems (Q) for which all of the surfaces $w = const.$ are ruled, and the determination of the intrinsic equations of such systems requires the integration of a system of partial differential equations of the first order.

In fact, for the case of the hyperboloid (1), we have

$$
D = 0, \qquad D' = -2\frac{\sqrt{H}}{\rho}, \qquad D'' = 2(u+v)^2\sqrt{H}\varphi, \tag{60}
$$

* B., chapter 3.

where φ is independent of u. Now equations (14) reduce to

$$\frac{\partial \theta}{\partial u} = 0, \qquad \frac{\partial \theta}{\partial v} = \frac{a'b'c'}{k} V\varphi; \tag{61}$$

the first of equations (35) may be replaced by

$$T = \log \frac{(u+v)^2}{\rho m} f, \tag{67}$$

where f is independent of u and the second of (35) requires that f satisfy

$$\frac{\partial f}{\partial v} = f \frac{V'}{V}. \tag{68}$$

The first two of equations (52) are satisfied identically, and the last requires that φ in (60) be a solution of

$$\frac{\partial \varphi}{\partial w} + \frac{2abc}{k} f \frac{\partial \varphi}{\partial v} + \sigma f \varphi = 0, \tag{69}$$

where σ is given by

$$\sigma = \left(\frac{F}{m} + \frac{G}{l}\right)\frac{(u+v)^2}{\rho} - \frac{L_0}{lm^2}\rho A\,(u+v)^2 + \frac{2abc}{k}\left(\frac{2}{u+v} + \frac{V'}{V}\right). \tag{69'}$$

It can readily be shown that each set of solutions of equations (61), (68) and (69) gives a system (Q) of the kind sought.

§14. *Conjugate Systems. Transformation of Systems* (Q).

If the equation

$$\bar{x} = x + l\frac{\partial x}{\partial u} + m\frac{\partial x}{\partial v} \tag{70}$$

be differentiated, we obtain

$$\begin{aligned}
\frac{\partial \bar{x}}{\partial u} &= L_0 \frac{\partial x}{\partial u} + M_0 \frac{\partial x}{\partial v} + \left(\frac{\partial l}{\partial \theta}\frac{\partial x}{\partial u} + \frac{\partial m}{\partial \theta}\frac{\partial x}{\partial v}\right)\frac{\partial \theta}{\partial u} + (Dl + D'm)X_3, \\
\frac{\partial \bar{x}}{\partial v} &= P_0 \frac{\partial x}{\partial u} + Q_0 \frac{\partial x}{\partial v} + \left(\frac{\partial l}{\partial \theta}\frac{\partial x}{\partial u} + \frac{\partial m}{\partial \theta}\frac{\partial x}{\partial v}\right)\frac{\partial \theta}{\partial v} + (D'l + D''m)X_3.
\end{aligned} \tag{71}$$

The affine transformation of Ivory which establishes the correspondence of applicability of S and $\bar{S}$ is given analytically by*

$$\bar{u} = \frac{\dfrac{b}{b'}\left[\dfrac{a}{a'}(1-uv) - \dfrac{c}{c'}(1+uv)\right](1+\sin\theta) + \dfrac{ac}{a'c'}\left[(u-v) + \dfrac{b}{b'}(u+v)\right]\cos\theta}{\dfrac{b}{b'}\left[\dfrac{a}{a'}(1-uv) + \dfrac{c}{c'}(1+uv)\right]\cos\theta + \dfrac{ac}{a'c'}\left[(u-v) - \dfrac{b}{b'}(u+v)\right](1+\sin\theta)}, \tag{72}$$

$$\bar{v} = \frac{1-\sin\theta}{\cos\theta}.$$

* B., p. 60.

From (71) and (72) we have accordingly

$$\frac{\partial \bar{x}}{\partial \bar{u}} = \frac{(u+v)\,a'\,b'\,c'\,\psi^2}{2Wabc\,(1+\sin\theta)}\left[\frac{L_0}{m}\frac{\partial x}{\partial u} + \frac{M_0}{m}\frac{\partial x}{\partial v} - \frac{2\epsilon\sqrt{abc}}{(u+v)^2\sqrt{\rho}}X_3\right],$$

$$(77)\quad \frac{\partial \bar{x}}{\partial \bar{v}} = -(1+\sin\theta)\left[\left(\frac{J}{m} - \frac{L_0 k\rho\,(u+v)^2}{2a'\,b'\,c'\,V}\right)\frac{\partial x}{\partial u}\right.$$

$$\left. + \left(\frac{J}{l} - \frac{M_0 k\rho\,(u+v)^2}{2a'\,b'\,c'\,V}\right)\frac{\partial x}{\partial v} + \frac{\epsilon\,(u+v)\,k\sqrt{abc}\,\sqrt{\rho}}{Wa'\,b'\,c'}X_3\right],$$

where ψ denotes the denominator of $\bar{u}$ in (72).

Proceeding as in § 7 we find that

$$(84)\quad J = -\frac{2abc\,(u+v)\,lm}{a'\,b'\,c'\,W},$$

and we establish the identity (50) in the form

$$(85)\quad El^2 + 2Flm + Gm^2 + \rho^2 A^2 = -\frac{4abclm\rho}{k\,(u+v)^2}.$$

Furthermore it can be shown that (70) may be written in the form

$$(92)\quad x = \bar{x} + \bar{l}\frac{\partial \bar{x}}{\partial \bar{u}} + \bar{m}\frac{\partial \bar{x}}{\partial \bar{v}},$$

where

$$(94)\quad \bar{l} = \frac{(1+\sin\theta)\,k\rho W\,(u+v)}{2a'\,b'\,c'\,\psi^2}, \qquad \bar{m} = -\frac{a'\,b'\,c'\,W}{2abc\,(1+\sin\theta)\,(u+v)}.$$

From these expressions we have

$$(95)\quad \rho = -\frac{4abc\bar{l}\bar{m}}{k\,(\bar{u}+\bar{v})^2}$$

which is analogous to the expression

$$(83)\quad \bar{\rho} = -\frac{4abclm}{k\,(u+v)^2},$$

arising in the proof of (85).

One shows readily that (72) is reciprocal in form, and by means of the expressions derived therefrom for u and v in terms of $\bar{u}$ and $\bar{v}$ it can be proved that

If $\bar{S}$ is a transform of S by a transformation B_k, S may be obtained from $\bar{S}$ by another $\bar{B}_k$, which is the inverse of the former.

When a system (Q) is transformed by a B_k, whose equations are

$$(96)\quad x_1 = x + l_1\frac{\partial x}{\partial u} + m_1\frac{\partial x}{\partial v},$$

where l_1 and m_1 denote the result of replacing θ and k in l and m by θ_1 and k_1, we obtain a system of surfaces S_1 applicable to the same hyperboloid. If each surface S_1 undergoes a transformation of parameters to a new set u_1 and v_1, analogous to $\bar{u}$ and $\bar{v}$, defined by (72), we have space referred to a triple system of surfaces of parameters u_1, v_1, w.

As shown in § 9, the direction of the tangent to the curves of parameter w are given by

$$\frac{\partial x_1}{\partial w} = X_1\sqrt{E}e^{\tau}[F(l_1-l)+G(m_1-m)]$$

$$(98)\qquad -X_2\sqrt{G}e^{\tau}[E(l_1-l)+F(m_1-m)]$$

$$+X_3\left[e^{\tau}\frac{2\epsilon\rho^{\frac{3}{2}}\sqrt{abc}A}{(u+v)^2}+l_1B+m_1C-\frac{2\epsilon_1k_1(u+v)\sqrt{\rho}\sqrt{abc}}{a_1'b_1'c_1'W_1}\frac{\partial\theta_1}{\partial w}\right]$$

When we take (92) in place of (96), this reduces to

$$(99)\qquad \frac{\partial\bar{x}}{\partial w}=2X_3\left[\frac{abc\rho e^{\tau}}{k(u+v)^2}(Dl^2-D''m^2)-\frac{\epsilon k(u+v)\sqrt{\rho}\sqrt{abc}}{a'b'c'W}\frac{\partial\theta}{\partial w}\right].$$

Hence the tangents to the curves of parameter w are parallel to the corresponding normals to the surfaces S, and consequently, if we say that the surfaces $\bar{S}$ form a system *conjugate* to the given system, we have the theorem:

The system conjugate to a given system (Q) *is a system* (Q).

As regards the existence of a generalized transformation of a system (Q) into a system (Q) the situation is similar to that set forth in § 10. However, in the next section we give an example of particular systems (Q) for which such transformations have been established.

§ 15. *Isogonal Deformations of Pseudospherical Surfaces.*

If in (1) we put*

$$(109)\qquad a=b=i,\qquad c=1$$

we have that Q is the imaginary sphere $x^2+y^2+z^2+1=0$, and consequently S is a pseudospherical surface. In this case we have

$$(110)\quad E=G=0,\qquad F=\frac{2}{(u+v)^2},\qquad H=\frac{-1}{(u+v)^4},\qquad \rho=1.$$

* Cf. B., §§ 49, 50.

Taking

$$k = \cos^2\sigma, \tag{111}$$

we find

$$a' = b' = i\sin\sigma, \qquad c' = \sin\sigma,$$

$$l = \frac{1-\sin\sigma}{2}(u+v)\frac{\lambda}{\mu}, \qquad m = \frac{1+\sin\sigma}{2}(u+v)\frac{\mu}{\lambda}, \tag{112}$$

$$W = \frac{2}{\sin^2\sigma}\,\frac{\lambda\mu}{1+\sin\theta},$$

where

$$\lambda = \cos\theta + u(1+\sin\theta), \qquad \mu = \cos\theta - v(1+\sin\theta). \tag{113}$$

If we put

$$u = \alpha + i\beta, \qquad v = \alpha - i\beta, \qquad e^{i\phi} = \frac{1-\sin\sigma}{\cos\sigma}\,\frac{\lambda}{\mu}, \tag{114}$$

we have for the new fundamental coefficients of the surface

$$E' = G' = \frac{1}{\alpha^2}, \qquad F' = 0,$$

$$\Delta = 2D' + D + D'', \quad \Delta' = i(D - D''), \quad \Delta'' = 2D' - D - D''. \tag{115}$$

It follows that

$$l = \alpha\cos\sigma e^{i\phi}, \qquad m = \alpha\cos\sigma e^{-i\phi}, \qquad A = -\sin\sigma. \tag{116}$$

Equations (14) become

$$\begin{aligned}\frac{\partial\varphi}{\partial\alpha} &= \frac{\sin\varphi}{\alpha\cos\sigma} + \alpha\tan\sigma\,(\Delta\cos\varphi + \Delta'\sin\varphi),\\ \frac{\partial\varphi}{\partial\beta} &= \frac{\cos\sigma - \cos\varphi}{\alpha\cos\sigma} + \alpha\tan\sigma\,(\Delta'\cos\varphi + \Delta''\sin\varphi).\end{aligned} \tag{117}$$

Furthermore (39) may be replaced by

$$\begin{aligned}\frac{\partial x}{\partial\alpha} &= \frac{X_1'}{\alpha}, \qquad \frac{\partial x}{\partial\beta} = \frac{X_2'}{\beta},\\ \frac{\partial x}{\partial w} &= h[\cos\sigma\,(\sin\varphi\,X_1 - \cos\varphi\,X_2) + \sin\sigma X_3],\end{aligned} \tag{118}$$

where

$$h = -\frac{2ie^T}{(u+v)^2} = -\frac{ie^T}{2\alpha^2}. \tag{119}$$

Equations (118) define a system (Q) such that the surfaces $w = \text{const.}$ are pseudospherical and the curves of parameter w are isogonal trajectories of these surfaces. These are the systems which Bianchi has discovered and, in fact, the equations are in the form given by him.*

For this case the equation analogous to (103), Part I, becomes

$$h_{33} = \frac{\alpha^2}{\cos^2\sigma}(h_{13}^2 + h_{23}^2), \tag{120}$$

when the linear element of space is written

$$ds^2 = \frac{d\alpha^2 + d\beta^2}{\alpha^2} + 2h_{13}\,d\alpha dw + 2h_{23}\,d\beta dw + h_{33}\,dw^2. \tag{121}$$

Bianchi* shows that the relation (120) characterizes systems (Q) for which the curves of parameter w are isogonal trajectories of the pseudospherical surfaces $w = \text{const.}$ When Bianchi expresses the condition (120) for the system arising from a given system (Q) by means of a transformation B_{σ_1}, he finds that φ_1 must satisfy in addition to (117), in which σ is replaced by σ_1, the further equation

$$\begin{aligned}\frac{\partial\varphi_1}{\partial w} &= \frac{\alpha\tan\sigma_1}{\sin\sigma}\cos\varphi_1\frac{\partial h}{\partial u} + \frac{u\tan\sigma_1}{\sin\sigma}\frac{\partial h}{\partial v} \\ &\quad + \frac{h\sin\sigma_1}{\sin\sigma - \sin\sigma_1} - \frac{h\cos\sigma\sin^2\sigma_1}{\sin\sigma\cos\sigma_1(\sin\sigma - \sin\sigma_1)}\cos(\varphi_1 - \varphi).\end{aligned} \tag{122}$$

Because of the particular form of the functions Bianchi was enabled to factor the equation analogous to (104) § 10, but we have indicated a method by which π can be formed, even if the factors are not apparent.

Thus we have a particular case in which a system (Q) of surfaces applicable to a central quadric is transformable into another system (Q). This strengthens the belief stated at the end of the preceding section. Furthermore, we are of the opinion that (104) is the sufficient as well as necessary condition that a family of continuous deforms of a central quadric forms a system (Q).

It should be added here that by suitable changes in constants and variables it is possible to transform the results of the preceding sections so that one shall obtain similar theorems for surfaces applicable to any central quadric.

* L. c., p. 20.

§ 16. *Deformation of the Hyperboloid of Revolution and of Bertrand Curves.*

We shall close our discussion with the consideration of the continuous deformation of a ruled-surface applicable to an hyperboloid of revolution, because one derives therefrom a continuous array of applicable curves of Bertrand.

Let S be a ruled surface applicable to an hyperboloid of revolution, then the line of striction is a Bertrand curve and corresponds to the circle of gorge on the hyperboloid. Referring to (1), we see that in the present case $b = a$ and the line of striction is given by $uv - 1 = 0$. Accordingly we effect the change of parameters

$$u = \frac{1-t}{v_1}, \qquad v = v_1,$$

so that the curve $t = 0$ on each surface S is the line of striction.

In this case

$$E_1 = \Sigma\left(\frac{\partial x}{\partial t}\right)^2 = \frac{E}{v^2}, \qquad F_1 = \Sigma \frac{\partial x}{\partial t}\frac{\partial x}{\partial v_1} = \frac{u}{v^2}E - \frac{1}{v}F, \quad G_1 = G - \frac{2u}{v}F + \frac{u^2}{v^2}E.$$

The direction-cosines of the tangent to the curve $t = 0$ are

$$\alpha, \beta, \gamma = \frac{\sqrt{a^2+c^2}}{2a}[(X_2 - X_1), \quad (Y_2 - Y_1), \quad (Z_2 - Z_1)],$$

where $X_1, \cdots, Z_2$ are given by (39). The direction-cosines of the principal normal are X_3, Y_3, Z_3 and of the binomial

$$\lambda, \mu, \nu = \frac{\sqrt{a^2+c^2}}{2c}[X_2 + X_1, Y_2 + Y_1, Z_2 + Z_1].$$

We denote by $\bar{S}$ the locus of the Bertrand curves $t = 0$ on the surfaces $w = \text{const.}$ of a system (Q) of the kind considered. The direction-cosines of the normal to $\bar{S}$ are of the form

$$\bar{X} = r[c\sqrt{a^2+c^2}\,A\lambda - a(\sqrt{E}l - \sqrt{G}m)X_3],$$

where

$$r = \sqrt{(\sqrt{E}l + \sqrt{G}m)^2 a^2 + c^2(a^2+c^2)A^2}.$$

If $\bar{\omega}$ denotes the angle between the principal normal to $t = 0$ and the normal to the surface, then

$$\cos\bar{\omega} = ar(\sqrt{G}m - \sqrt{E}l), \qquad \sin\bar{\omega} = crA\sqrt{a^2+c^2}.$$

One finds without difficulty the first and second fundamental coefficients of $\bar{S}$, and therefrom obtains

$$\frac{\cos\bar{\omega}}{\rho_0}=\frac{\bar{D}}{\bar{E}}=\frac{1}{2}[\,c\,(1+v^2)^2\,\varphi+2\,](\,\sqrt{G}m-\sqrt{E}l\,)\,r\,,$$

$$\frac{1}{T}=\frac{\bar{F}\bar{D}-\bar{E}\bar{D}'}{\bar{E}\sqrt{\bar{E}\bar{G}-\bar{F}^2}}=-\,cr^2\,(a^2+c^2)\,A$$

$$+\,r^2\sqrt{a^2+c^2}\left[\frac{e^{-T}}{4acv^2}\left(C-\frac{B}{v^2}\right)-\frac{(lv^2+m)c}{(1+v^2)^2}\left(c\varphi+\frac{2}{(1+v^2)^2}\right)\right]$$

$$(\,\sqrt{E}l-\sqrt{G}m)\,(1+v^2)^3\,,$$

where ρ_0 and T are the radii of first curvature and geodesic torsion respectively of the curve $t=0$ and B and C are the functions defined by (41). Since the torsion is given by

$$\frac{1}{\tau}=\frac{1}{T}+\frac{d\bar{\omega}}{ds_v}\,,$$

we are in a position to find τ. By easy reduction it is found that

$$\frac{1}{\tau}=\frac{1}{2}(1+v^2)\varphi\,.$$

Since the element of any of the curve is given by $\sqrt{E}\,dv_1$, one finds readily that

$$v=\tan\frac{s}{2a}\,.$$

Hence the intrinsic equations of the curve are

$$\frac{1}{\rho_0}=\frac{1}{2a}\left(c\varphi\sec^4\frac{s}{2a}+2\right),\qquad \frac{1}{\tau}=\frac{\varphi}{2}\sec^4\frac{s}{2a}, \tag{123}$$

where from § 13 it follows that φ satisfies the equations

$$\frac{\partial\theta}{\partial s}=\frac{(a^2+k)\sqrt{c^2-k}\,2aV\varphi}{k\,(s^2+4a^2)},\qquad \frac{\partial f}{\partial s}=\frac{f}{V}\frac{\partial V}{\partial v}\cdot\frac{2a}{s^2+4a^2},$$

$$\frac{\partial\varphi}{\partial w}+\frac{ac}{k}f\,(s^2+4a^2)\frac{\partial\varphi}{\partial s}+\sigma f\varphi=0\,, \tag{124}$$

σ being a function of s and θ obtained by replacing u and v in (69′) by $\cot s/2a$ and $\tan s/2a$ respectively.

The surfaces $w = \text{const.}$ of the system (Q) conjugate to the given system are likewise applicable to the same hyperboloid. Moreover to a curve of Bertrand of the original system corresponds a curve of Bertrand of the second system in such a way that the join of corresponding points is tangent to the two surfaces of conjugate systems; we call the second curve the *conjugate Bertrand curve*. Hence we have the theorem:

To each function φ satisfying equations (124) *there corresponds a family of curves of Bertrand defined by* (123) *such that all of these curves are continuous deforms of one another, the parameter of deformation being w, the direction of deformation being parallel to the principal normal of the conjugate curve at the corresponding point.*

Later we shall consider general deformations of Bertrand curves.

PRINCETON,
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